



**QUEEN'S
UNIVERSITY
BELFAST**

Operator systems from discrete groups

Farenick, D., Kavruk, A. S., Paulsen, V. I., & Todorov, I. G. (2014). Operator systems from discrete groups. *Communications in Mathematical Physics*, 329(1), 207-238. <https://doi.org/10.1007/s00220-014-2037-6>

Published in:
Communications in Mathematical Physics

Document Version:
Peer reviewed version

Queen's University Belfast - Research Portal:
[Link to publication record in Queen's University Belfast Research Portal](#)

Publisher rights
The final publication is available at Springer via <http://dx.doi.org/10.1007/s00220-014-2037-6>

General rights
Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy
The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact openaccess@qub.ac.uk.

OPERATOR SYSTEMS FROM DISCRETE GROUPS

DOUGLAS FARENICK, ALI S. KAVRUK, VERN I. PAULSEN, AND IVAN G. TODOROV

ABSTRACT. We formulate a general framework for the study of operator systems arising from discrete groups. We study in detail the operator system of the free group \mathbb{F}_n on n generators, as well as the operator systems of the free products of finitely many copies of the two-element group \mathbb{Z}_2 . We examine various tensor products of group operator systems, including the minimal, the maximal, and the commuting tensor products. We introduce a new tensor product in the category of operator systems and formulate necessary and sufficient conditions for its equality to the commuting tensor product in the case of group operator systems. We express various sets of quantum correlations studied in the theoretical physics literature in terms of different tensor products of operator systems of discrete groups. We thus recover earlier results of Tsirelson and formulate a new approach for the study of quantum correlation boxes.

INTRODUCTION

In this paper we study operator systems arising from discrete groups. Let \mathbf{u} be a set of generators of a discrete group G . The operator system of \mathbf{u} is defined to be the operator subsystem $\mathcal{S}(\mathbf{u})$ of the (full) group C^* -algebra $C^*(G)$ given by

$$(1) \quad \mathcal{S}(\mathbf{u}) = \text{span}\{1, u, u^* : u \in \mathbf{u}\} \subseteq C^*(G).$$

We will show that many “universal operator systems” arise in this manner. For example, the universal operator systems of n unitary operators or of n contractive operators arise from the study of the operator system $\mathcal{S}(\mathbf{u})$ obtained by letting $\mathbf{u} = \{u_1, \dots, u_n\}$ be the set of standard generators of the free group \mathbb{F}_n on n generators (see Section 4).

We show that a number of questions studied previously in the literature can be placed in the framework of tensor products of group operator systems. For example, in our earlier work [7, 6, 11] it has been established that Connes’ Embedding Problem, in the guise of Kirchberg’s Problem, reduces to understanding the “minimal” and the “commuting” tensor products [14] of some group operator systems. Specifically, if we let \mathcal{S}_n denote the operator subsystem of $C^*(\mathbb{F}_n)$ obtained as in (1) using the set \mathbf{u} of the canonical generators for the free group \mathbb{F}_n , then the following statements are equivalent:

- (i) $\mathcal{S}_2 \otimes_{\min} \mathcal{S}_2 = \mathcal{S}_2 \otimes_c \mathcal{S}_2$;
- (ii) $C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2)$;
- (iii) the Connes Embedding Problem has an affirmative solution.

2010 *Mathematics Subject Classification.* Primary 46L06, 46L07; Secondary 46L05, 47L25, 47L90.

Key words and phrases. operator system, tensor product, discrete group, free group, free product, Tsirelson’s problem, quantum correlations.

This work supported in part by NSERC (Canada), NSF (USA), and EPSRC (United Kingdom).

The equivalence of the last two assertions above is the criterion of Kirchberg [15]; however, statement (i) involves a vector space of dimension 9 and is, on the surface, seemingly more tractable than the Kirchberg criterion.

Because of their “universal” and “free” properties, there are certain tensor identities that hold for tensor products of the form $\mathcal{S}_n \otimes \mathcal{S}_m$ that are not apparent for other group operator systems. For this reason we are forced to introduce a new tensor product, which we denote by \otimes_{ess} . It complements the list of tensor product introduced in [14] and plays an important role in the study of group operator systems. In Section 2 we determine necessary and sufficient conditions for the equality of the \otimes_{ess} and the commuting tensor products in the case of group operator systems.

The problem of whether $\mathcal{S}_2 \otimes_{\min} \mathcal{S}_2 = \mathcal{S}_2 \otimes_c \mathcal{S}_2$ is equivalent to the Connes Embedding Problem. What is the relationship between $\mathcal{S}_2 \otimes_c \mathcal{S}_2$ and $\mathcal{S}_2 \otimes_{\max} \mathcal{S}_2$? In Theorem 3.8 we show that $\mathcal{S}_n \otimes_c \mathcal{S}_m \neq \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$ for every $n, m \in \mathbb{N}$ and that the distinction between them occurs at the level of 2×2 matrices over $\mathcal{S}_n \otimes_c \mathcal{S}_m$ and $\mathcal{S}_n \otimes_{\max} \mathcal{S}_m$. One interesting consequence: although the C^* -envelope $C_e^*(\mathcal{S}_k)$ of \mathcal{S}_k is $C^*(\mathbb{F}_k)$, it is not true that $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m) = C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_m)$ (Theorem 3.9).

The study of \mathcal{S}_n in the case where $n = 1$ is of interest in operator theory, as the operator system \mathcal{S}_1 and its dual \mathcal{S}_1^d are universal objects for operators of a certain type, as we shall explain in Section 4. A complete list of tensor product relations between \mathcal{S}_1 and \mathcal{S}_1^d is given in Theorem 4.4 for the min, c, and max tensor products.

There are further reasons for undertaking a detailed study of tensor products of operator systems arising from discrete groups. As suggested, for example, in [8], operator system tensor products offer a useful framework for the description of quantum correlations. However, in [8] only the ordered structure is used fully and the matrix orderings are not exploited or characterized. In the present paper, we make explicit the connections between quantum correlations and tensor products of group operator systems. In this regard, we show in Section 5 that the “non-commutative n -cube” studied by Tsirelson [19, 20] in connection with quantum generalisations of the Bell inequalities generates an operator system, which we denote by $NC(n)$, that is also the operator system arising from the n -fold free product $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$. In Theorem 6.13 we establish an operator system interpretation of Tsirelson’s computations in [19, 20]: namely, $NC(n) \otimes_c NC(m) \neq NC(n) \otimes_{\max} NC(m)$ for all $n, m \geq 2$.

Finally, in Section 7 we examine bipartite correlations from the perspective of three operator system structures on the algebraic tensor product $NC(2)^d \otimes NC(2)^d$, where $NC(2)^d$ is given a concrete realisation as an operator subsystem of 4×4 diagonal matrices.

1. TENSOR PRODUCTS AND QUOTIENTS OF OPERATOR SYSTEMS

In this section, we introduce basic terminology and notation, and recall previous constructions and results that will be needed in the sequel. If V is a vector space, we let $\mathcal{M}_{n,m}(V)$ be the space of all n by m matrices with entries in V . We set $\mathcal{M}_n(V) = \mathcal{M}_{n,n}(V)$ and $\mathcal{M}_n = \mathcal{M}_n(\mathbb{C})$. We let $(E_{ij})_{i,j}$ be the canonical matrix unit system in \mathcal{M}_n . For a map $\phi : V \rightarrow W$ between vector spaces, we let $\phi^{(n)} : \mathcal{M}_n(V) \rightarrow \mathcal{M}_n(W)$ be the n th ampliation of ϕ given by $\phi^{(n)}((x_{ij})_{i,j}) = (\phi(x_{i,j}))_{i,j}$. For a Hilbert space \mathcal{H} , we denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . An *operator system* is a subspace \mathcal{S} of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} which contains

the identity operator I and is closed under taking adjoints. The embedding of $\mathcal{M}_n(\mathcal{S})$ into $\mathcal{B}(\mathcal{H}^n)$ gives rise to the cone $\mathcal{M}_n(\mathcal{S})_+$ of all positive operators in $\mathcal{M}_n(\mathcal{S})$. The family $(\mathcal{M}_n(\mathcal{S})_+)_{n \in \mathbb{N}}$ of cones is called the *operator system structure* of \mathcal{S} . Every complex $*$ -vector space equipped with a family of matricial cones and an order unit satisfying natural axioms can, by virtue of the Choi-Effros Theorem [4], be represented faithfully as an operator system acting on some Hilbert space. When a particular embedding is not specified, the order unit of an operator system will be denoted by 1. A map $\phi : \mathcal{S} \rightarrow \mathcal{T}$ between operator systems is called *completely positive* if $\phi^{(n)}$ is positive, that is, $\phi^{(n)}(\mathcal{M}_n(\mathcal{S})_+) \subseteq \mathcal{M}_n(\mathcal{T})_+$, for every $n \in \mathbb{N}$. A linear bijection $\phi : \mathcal{S} \rightarrow \mathcal{T}$ of operator systems \mathcal{S} and \mathcal{T} is a *complete order isomorphism* if both ϕ and ϕ^{-1} are completely positive. We refer the reader to [17] for further properties of operator systems and completely positive maps.

An *operator system tensor product* $\mathcal{S} \otimes_{\tau} \mathcal{T}$ of operator systems \mathcal{S} and \mathcal{T} is an operator system structure on the algebraic tensor product $\mathcal{S} \otimes \mathcal{T}$ satisfying a set of natural axioms. We refer the reader to [14], where a detailed study of such tensor products was undertaken. Suppose that $\mathcal{S}_1 \subseteq \mathcal{T}_1$ and $\mathcal{S}_2 \subseteq \mathcal{T}_2$ are inclusions of operator systems. Let $\iota_j : \mathcal{S}_j \rightarrow \mathcal{T}_j$ denote the inclusion maps $\iota_j(x_j) = x_j$ for $x_j \in \mathcal{S}_j$, $j = 1, 2$, so that the map $\iota_1 \otimes \iota_2 : \mathcal{S}_1 \otimes \mathcal{S}_2 \rightarrow \mathcal{T}_1 \otimes \mathcal{T}_2$ is a linear inclusion of vector spaces. If τ and σ are operator system structures on $\mathcal{S}_1 \otimes \mathcal{S}_2$ and $\mathcal{T}_1 \otimes \mathcal{T}_2$ respectively, then we use the notation

$$\mathcal{S}_1 \otimes_{\tau} \mathcal{S}_2 \subseteq^+ \mathcal{T}_1 \otimes_{\sigma} \mathcal{T}_2$$

to denote that $\iota_1 \otimes \iota_2 : \mathcal{S}_1 \otimes_{\tau} \mathcal{S}_2 \rightarrow \mathcal{T}_1 \otimes_{\sigma} \mathcal{T}_2$ is a (unital) completely positive map. This notation is motivated by the fact that $\iota_1 \otimes \iota_2$ is a completely positive map if and only if, for every n , the cone $\mathcal{M}_n(\mathcal{S}_1 \otimes_{\tau} \mathcal{S}_2)_+$ is contained in the cone $\mathcal{M}_n(\mathcal{T}_1 \otimes_{\sigma} \mathcal{T}_2)_+$. If, in addition, $\iota_1 \otimes \iota_2$ is a complete order isomorphism onto its range, then we write

$$\mathcal{S}_1 \otimes_{\tau} \mathcal{S}_2 \subseteq_{\text{coi}} \mathcal{T}_1 \otimes_{\sigma} \mathcal{T}_2.$$

In particular, if τ and σ are two operator system structures on $\mathcal{S} \otimes \mathcal{T}$, then

$$\mathcal{S} \otimes_{\tau} \mathcal{T} = \mathcal{S} \otimes_{\sigma} \mathcal{T} \quad \text{means} \quad \mathcal{S} \otimes_{\tau} \mathcal{T} \subseteq_{\text{coi}} \mathcal{S} \otimes_{\sigma} \mathcal{T} \quad \text{and} \quad \mathcal{S} \otimes_{\sigma} \mathcal{T} \subseteq_{\text{coi}} \mathcal{S} \otimes_{\tau} \mathcal{T}.$$

When $\mathcal{S}_1 \otimes_{\tau} \mathcal{S}_2 \subseteq^+ \mathcal{S}_1 \otimes_{\sigma} \mathcal{S}_2$, then we will also write $\tau \geq \sigma$ and say that τ majorizes σ .

In the sequel, we will use extensively the following operator system tensor products introduced in [14]:

(a) *The minimal tensor product* min. If $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{T} \subseteq \mathcal{B}(\mathcal{K})$, where \mathcal{H} and \mathcal{K} are Hilbert spaces, then $\mathcal{S} \otimes_{\min} \mathcal{T}$ is the operator system arising from the natural inclusion of $\mathcal{S} \otimes \mathcal{T}$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

(b) *The maximal tensor product* max. For each $n \in \mathbb{N}$, let $D_n = \{A^*(P \otimes Q)A : A \in \mathcal{M}_{n,km}(\mathbb{C}), P \in \mathcal{M}_k(\mathcal{S})_+, Q \in \mathcal{M}_m(\mathcal{T})_+\}$. The Archimedeanization [18] of the family $(D_n)_{n \in \mathbb{N}}$ of cones is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$; the corresponding operator system is denoted by $\mathcal{S} \otimes_{\max} \mathcal{T}$.

(c) *The commuting tensor product* c. By definition, an element $X \in \mathcal{M}_n(\mathcal{S} \otimes \mathcal{T})$ belongs to the positive cone $\mathcal{M}_n(\mathcal{S} \otimes_c \mathcal{T})_+$ if $(\phi \cdot \psi)^{(n)}(X)$ is a positive operator for all completely positive maps $\phi : \mathcal{S} \rightarrow \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ with commuting ranges. Here, the linear map $\phi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \rightarrow \mathcal{B}(\mathcal{H})$ is given by $\phi \cdot \psi(x \otimes y) = \phi(x)\psi(y)$, $x \in \mathcal{S}$, $y \in \mathcal{T}$.

The tensor products \min , c , and \max are functorial in the sense that if τ denotes any of them, and $\phi : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ and $\psi : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ are completely positive maps, then the tensor product map $\phi \otimes \psi : \mathcal{S}_1 \otimes_\tau \mathcal{T}_1 \rightarrow \mathcal{S}_2 \otimes_\tau \mathcal{T}_2$ is completely positive.

A new tensor product ess will be introduced in the next section. This tensor product and the three tensor products mentioned above satisfy the relations

$$\mathcal{S} \otimes_{\max} \mathcal{T} \subseteq^+ \mathcal{S} \otimes_c \mathcal{T} \subseteq^+ \mathcal{S} \otimes_{\text{ess}} \mathcal{T} \subseteq^+ \mathcal{S} \otimes_{\min} \mathcal{T}$$

for all operator systems \mathcal{S} and \mathcal{T} .

For every operator system \mathcal{S} , we denote by \mathcal{S}^d the (normed space) dual of \mathcal{S} . The space $\mathcal{M}_n(\mathcal{S}^d)$ can be naturally identified with a subspace of the space $\mathcal{L}(\mathcal{S}, \mathcal{M}_n)$ of all linear maps from \mathcal{S} into \mathcal{M}_n . Taking the pre-image of the cone of all completely positive maps in $\mathcal{L}(\mathcal{S}, \mathcal{M}_n)$, we obtain a family of matricial cones on \mathcal{S}^d , which turns it into an operator system. We have, in particular, that \mathcal{S}_+^d consists of all positive functionals on \mathcal{S} ; the elements $\phi \in \mathcal{S}_+^d$ with $\phi(1) = 1$ are called *states* of \mathcal{S} . An important case arises when \mathcal{S} is finite dimensional; in this case, \mathcal{S}^d is an operator system when equipped with the matricial cones family just described and an order unit is given by any faithful state on \mathcal{S} [4, Corollary 4.5].

We now move to the notion of quotients in the operator system category.

Definition 1.1. *A linear subspace $\mathcal{J} \subseteq \mathcal{S}$ of an operator system \mathcal{S} is called a kernel if there is an operator system \mathcal{T} and a completely positive linear map $\phi : \mathcal{S} \rightarrow \mathcal{T}$ such that $\mathcal{J} = \ker \phi$.*

If $\mathcal{J} \subseteq \mathcal{S}$ is kernel, then one may endow the $*$ -vector space \mathcal{S}/\mathcal{J} with an operator system structure such that the canonical quotient map $q_{\mathcal{J}} : \mathcal{S} \rightarrow \mathcal{S}/\mathcal{J}$ is unital and completely positive [13]. Moreover, if $\mathcal{J} \subseteq \ker \phi$ for some completely positive map $\phi : \mathcal{S} \rightarrow \mathcal{T}$, then there exists a completely positive map $\dot{\phi} : \mathcal{S}/\mathcal{J} \rightarrow \mathcal{T}$ such that $\phi = \dot{\phi} \circ q_{\mathcal{J}}$.

Definition 1.2. *A completely positive map $\phi : \mathcal{S} \rightarrow \mathcal{T}$ is called a complete quotient map if the natural quotient map $\dot{\phi} : \mathcal{S}/\ker \phi \rightarrow \mathcal{T}$ is a complete order isomorphism.*

The following complete quotient map will be used in the sequel.

Example 1.3. ([7]) Let u_1, \dots, u_{k-1} denote free unitary generators of $C^*(\mathbb{F}_{k-1})$ and set $\mathcal{S}_{k-1} \subset C^*(\mathbb{F}_{k-1})$ to be the span of $\{1, u_1, u_1^*, \dots, u_{k-1}, u_{k-1}^*\}$. Let \mathcal{T}_k operator subsystem of \mathcal{M}_k consisting of all tri-diagonal matrices and let $\{E_{ij}\}_{|i-j| \leq 1} \subset \mathcal{M}_k$ denote the matrix units that span \mathcal{T}_k . Then the linear map $\phi_k : \mathcal{T}_k \rightarrow \mathcal{S}_{k-1}$ for which

$$\phi_k(E_{ii}) = \frac{1}{k}1, \quad \phi_k(E_{j,j+1}) = \frac{1}{k}u_j, \quad \text{and} \quad \phi_k(E_{j+1,j}) = \frac{1}{k}u_j^*$$

for $i = 1, \dots, k$ and $j = 1, \dots, k-1$, is a complete quotient map.

We remark that the operator system quotient $\mathcal{T}_k/\ker \phi_k$ in Example 1.3 is in fact different from the operator space quotient $(\mathcal{T}_k/\ker \phi_k)_{\text{osp}}$ within the category of operator spaces [7].

To determine which surjective completely positive maps are complete quotient maps, there is a useful criterion based on strict positivity.

Definition 1.4. *An element h of an operator system \mathcal{S} is called strictly positive if there is a $\delta > 0$ such that $h - \delta 1_{\mathcal{S}} \in \mathcal{S}_+$.*

The following fact will be useful for us on several occasions.

Proposition 1.5. ([6, Proposition 3.2]) *A completely positive surjection $\phi : \mathcal{S} \rightarrow \mathcal{T}$ is a complete quotient map if and only if for every $p \in \mathbb{N}$ and every strictly positive element $Y \in \mathcal{M}_p(\mathcal{T})$ there is a strictly positive element $X \in \mathcal{M}_p(\mathcal{S})$ such that $Y = \phi^{(p)}(X)$.*

We record here for future reference two facts about positivity, the first of which is elementary and its proof is thus omitted.

Lemma 1.6. *Let \mathcal{T} be an operator system. If $A = [\alpha_{ij}]_{i,j} \in \mathcal{M}_p(\mathbb{C})_+$ and $T = [t_{ij}]_{i,j} \in \mathcal{M}_p(\mathcal{T})_+$, then $A \circ T := [\alpha_{ij}t_{ij}]_{i,j} \in \mathcal{M}_p(\mathcal{T})_+$.*

Lemma 1.7. *If $h \in \mathcal{S} \otimes_{\max} \mathcal{T}$ is strictly positive, then there exist $n \in \mathbb{N}$, $S = [s_{ij}] \in \mathcal{M}_n(\mathcal{S})_+$ and $T = [t_{ij}] \in \mathcal{M}_n(\mathcal{T})_+$ such that*

$$(2) \quad h = \sum_{i,j=1}^n s_{ij} \otimes t_{ij}.$$

Proof. By hypothesis, $h - \delta 1 \in (\mathcal{S} \otimes_{\max} \mathcal{T})_+$ for some real $\delta > 0$. Therefore, by the definition of the positive cone of $\mathcal{S} \otimes_{\max} \mathcal{T}$ [14, §5], there exist $n, m \in \mathbb{N}$, $P \in \mathcal{M}_n(\mathcal{S})_+$, $Q \in \mathcal{M}_m(\mathcal{T})_+$, and a linear map $\alpha : \mathbb{C} \rightarrow \mathbb{C}^{nm}$ such that

$$h = (h - \delta 1) + \delta 1 = \alpha^*(P \otimes Q)\alpha = \sum_{i,j=1}^n \sum_{k,\ell=1}^m \bar{\alpha}_{ik} p_{ij} \otimes q_{k\ell} \alpha_{j\ell},$$

where $[\alpha_{11}, \dots, \alpha_{1m}, \alpha_{21}, \dots, \alpha_{2m}, \dots] = \alpha$. For each pair i, j , set $s_{ij} = p_{ij}$ and $t_{ij} = \sum_{k,\ell=1}^m \bar{\alpha}_{ik} q_{k\ell} \alpha_{j\ell}$. Then, letting $A = [\alpha_{ji}]_{i,j}$, we have $[t_{ij}]_{i,j} = A^* T A$; thus, $[t_{ij}]_{i,j} \in \mathcal{M}_n(\mathcal{T})_+$ and formula (2) holds. \square

2. THE GENERAL FRAMEWORK

In this section, we describe the general framework for the study of operator systems of discrete groups. We recall that if G is a discrete group, then G embeds canonically into its full group C^* -algebra $C^*(G)$. The C^* -algebra $C^*(G)$ has the following universal property: for every unitary representation π of G on a Hilbert space \mathcal{H} , there exists a $*$ -representation $\tilde{\pi} : C^*(G) \rightarrow \mathcal{B}(\mathcal{H})$ such that $\tilde{\pi}(g) = \pi(g)$ for every $g \in G$.

Definition 2.1. *Let \mathbf{u} be a set of generators of a discrete group G . The operator system generated by \mathbf{u} is the operator subsystem $\mathcal{S}(\mathbf{u})$ of the group C^* -algebra $C^*(G)$ defined by*

$$\mathcal{S}(\mathbf{u}) = \text{span}\{1, u, u^* : u \in \mathbf{u}\}.$$

We recall two canonical C^* -covers of an operator system \mathcal{S} . The *universal C^* -algebra* $C_u^*(\mathcal{S})$ of \mathcal{S} [16] can be defined by the universal property that whenever $\phi : \mathcal{S} \rightarrow \mathcal{B}(\mathcal{H})$ is a unital completely positive map for some Hilbert space \mathcal{H} , there exists a (unique) $*$ -representation of $C_u^*(\mathcal{S})$ on \mathcal{H} extending ϕ . The *enveloping C^* -algebra* $C_e^*(\mathcal{S})$ of \mathcal{S} [9, 17] is defined by the universal property that whenever \mathcal{A} is a C^* -cover of \mathcal{S} , there exists a $*$ -homomorphism $\pi : \mathcal{A} \rightarrow C_e^*(\mathcal{S})$ such that $\pi(x) = x$ for every $x \in \mathcal{S}$.

Let ι_e and ι_u denote, respectively, the unital completely isometric embeddings of $\mathcal{S}(\mathbf{u})$ into $C_e^*(\mathcal{S}(\mathbf{u}))$ and $C_u^*(\mathcal{S}(\mathbf{u}))$, and let

$$\mathcal{R}(\mathbf{u}) = \iota_u(\mathcal{S}(\mathbf{u})).$$

Thus, $\mathcal{S}(\mathbf{u})$ and $\mathcal{R}(\mathbf{u})$ are completely order isomorphic operator systems and $C_u^*(\mathcal{S}(\mathbf{u}))$ is generated as a C^* -algebra by $\mathcal{R}(\mathbf{u})$. Hence, there are three, possibly distinct, C^* -algebras naturally affiliated with the operator system $\mathcal{S}(\mathbf{u})$:

- (1) the C^* -envelope $C_e^*(\mathcal{S}(\mathbf{u}))$ of $\mathcal{S}(\mathbf{u})$;
- (2) the C^* -algebra $C^*(G)$ generated by $\mathcal{S}(\mathbf{u})$, and
- (3) the universal C^* -algebra $C_u^*(\mathcal{S}(\mathbf{u}))$.

We note that there are a number of differences between these C^* -algebras. For example, each $u \in \mathcal{S}(\mathbf{u})$ is a unitary element of $C^*(G)$, but if $\tilde{u} = \iota_u(u)$, then the construction of $C_u^*(\mathcal{S}(\mathbf{u}))$ [16, Proposition 8] shows that \tilde{u} is never normal (unless G is the trivial group). In particular, \mathbf{u} is not a set of unitaries in $C_u^*(\mathcal{S}(\mathbf{u}))$. (As a concrete example, let $G = \mathbb{F}_1$, the free group on a single generator u . Then $C^*(\mathbb{F}_1)$ is $*$ -isomorphic via Fourier transform to the abelian C^* -algebra $C(\mathbb{T})$ of all complex valued continuous functions on the unit circle \mathbb{T} . On the other hand, $C_u^*(\mathcal{S}(\mathbf{u}))$ is nonabelian, but is singly generated; hence, the generator $\tilde{u} = \iota_u(u)$ is nonnormal.)

The situation with C^* -envelopes is more tractable. Indeed, the following proposition is a special case of [11, Proposition 5.6].

Proposition 2.2. *Up to a $*$ -isomorphism that fixes the elements of $\mathcal{S}(\mathbf{u})$, we have that $C_e^*(\mathcal{S}(\mathbf{u})) = C^*(G)$.*

Since the C^* -envelope of $\mathcal{S}(\mathbf{u})$ recaptures the group C^* -algebra, it is natural to ask for a description of the C^* -algebras arising from the tensor products of group operator systems. For arbitrary operator systems we have [14, Theorem 6.4]

$$\mathcal{S} \otimes_c \mathcal{T} \subseteq_{\text{coi}} C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}).$$

This leads us to the following definition:

Definition 2.3. *Given operator systems \mathcal{S} and \mathcal{T} , we let $\mathcal{S} \otimes_{\text{ess}} \mathcal{T}$ be the operator system defined by the inclusion*

$$\mathcal{S} \otimes_{\text{ess}} \mathcal{T} \subseteq_{\text{coi}} C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T}).$$

Since the images of \mathcal{S} and \mathcal{T} inside $C_e^*(\mathcal{S}) \otimes_{\max} C_e^*(\mathcal{T})$ under the canonical identifications commute, we have that $\mathcal{S} \otimes_c \mathcal{T} \subseteq^+ \mathcal{S} \otimes_{\text{ess}} \mathcal{T}$, i.e., $\text{ess} \leq c$. In the next section, we will prove that for free groups these two tensor products are identical. For the moment, we turn our attention to some conditions that guarantee the equality of ess with other tensor products.

Lemma 2.4. *Let G and H be discrete groups and $\mathbf{u} \subseteq G$ and $\mathbf{v} \subseteq H$ be finite generating sets. Let τ be an operator system tensor product and fix $u \in \mathbf{u}$ and $v \in \mathbf{v}$. If $u \otimes 1$ or $1 \otimes v$ is a unitary element of $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_{\tau} \mathcal{S}(\mathbf{v}))$, then $(u \otimes 1)(1 \otimes v) = (1 \otimes v)(u \otimes 1) = u \otimes v$.*

Proof. The argument is motivated by the ideas of [21]. Each of the matrices

$$p = \begin{bmatrix} 1 & u & u \\ u^* & 1 & 1 \\ u^* & 1 & 1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 & 1 & v \\ 1 & 1 & v \\ v^* & v^* & 1 \end{bmatrix}$$

is positive in $\mathcal{M}_3(\mathcal{S}(\mathbf{u}))$ and $\mathcal{M}_3(\mathcal{S}(\mathbf{v}))$, respectively. Therefore, $p \otimes q$ is positive in $\mathcal{M}_9(\mathcal{S}(\mathbf{u}) \otimes_{\max} \mathcal{S}(\mathbf{v}))$. Since $\tau \leq \max$, this element is also positive in $\mathcal{M}_9(\mathcal{S}(\mathbf{u}) \otimes_{\tau} \mathcal{S}(\mathbf{v}))$.

Let $A \in \mathcal{M}_3(\mathcal{C}_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v})))$ be given by

$$A = \left[\begin{array}{c|cc} 1 \otimes 1 & u \otimes 1 & u \otimes v \\ \hline u^* \otimes 1 & 1 \otimes 1 & 1 \otimes v \\ u^* \otimes v^* & 1 \otimes v^* & 1 \otimes 1 \end{array} \right] = \begin{bmatrix} 1 \otimes 1 & X \\ X^* & C \end{bmatrix}.$$

The matrix A is also expressed in block form by

$$A = \left[\begin{array}{cc|c} 1 \otimes 1 & u \otimes 1 & u \otimes v \\ \hline u^* \otimes 1 & 1 \otimes 1 & 1 \otimes v \\ \hline u^* \otimes v^* & 1 \otimes v^* & 1 \otimes 1 \end{array} \right] = \begin{bmatrix} D & Z \\ Z^* & 1 \otimes 1 \end{bmatrix}.$$

Because $A = \alpha(p \otimes q)\alpha^*$ for a suitable rectangular zero-one matrix α , the matrix A is a positive element of $\mathcal{M}_3(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$. By the Cholesky Algorithm, the matrix $\begin{bmatrix} 1 & X \\ X^* & C \end{bmatrix}$ is positive if and only if $C - X^*X$ is positive and the matrix $\begin{bmatrix} D & Z \\ Z^* & 1 \end{bmatrix}$ is positive if and only if $D - ZZ^*$ is positive. Hence,

$$(3) \quad 0 \leq C - X^*X = \begin{bmatrix} 1 - (u \otimes 1)^*(u \otimes 1) & 1 \otimes v - (u \otimes 1)^*(u \otimes v) \\ * & * \end{bmatrix}$$

and

$$(4) \quad 0 \leq D - ZZ^* = \begin{bmatrix} * & u \otimes 1 - (u \otimes v)(1 \otimes v)^* \\ * & 1 \otimes 1 - (1 \otimes v)(1 \otimes v)^* \end{bmatrix}.$$

Therefore, if $u \otimes 1$ is unitary, then inequality (3) holds only if $1 \otimes v = (u \otimes 1)^*(u \otimes v)$, and so $(u \otimes 1)(1 \otimes v) = u \otimes v$. Alternatively, if $1 \otimes v$ is unitary, then inequality (4) holds only if $u \otimes 1 = (u \otimes v)(1 \otimes v)^*$, which implies $(u \otimes 1)(1 \otimes v) = u \otimes v$. Hence, if $u \otimes 1$ or $1 \otimes v$ is unitary, then $(u \otimes 1)(1 \otimes v) = u \otimes v$.

Repeat this argument above using the matrices

$$\tilde{p} = \begin{bmatrix} 1 & 1 & u \\ 1 & 1 & u \\ u^* & u^* & 1 \end{bmatrix} \quad \text{and} \quad \tilde{q} = \begin{bmatrix} 1 & v & v \\ v^* & 1 & 1 \\ v^* & 1 & 1 \end{bmatrix}$$

in place of p and q to obtain $(1 \otimes v)(u \otimes 1) = u \otimes v$. \square

The theorem below characterises the situation in which the ambient C^* -algebra is not the maximal tensor products of universal C^* -algebras, but rather the maximal tensor product $\mathcal{C}_e^*(\mathcal{S}(\mathbf{u})) \otimes_{\max} \mathcal{C}_e^*(\mathcal{S}(\mathbf{v}))$ of enveloping C^* -algebras.

Theorem 2.5. *Let G and H be discrete groups and $\mathbf{u} \subseteq G$ and $\mathbf{v} \subseteq H$ be generating sets. Suppose that τ is an operator system tensor product such that $\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}) \subseteq^+ \mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v})$. Then the following statements are equivalent:*

- (1) $u \otimes 1$ and $1 \otimes v$ are unitary elements of $\mathcal{C}_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$ for every $u \in \mathbf{u}$ and $v \in \mathbf{v}$;
- (2) up to a $*$ -isomorphism, $\mathcal{C}_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v})) = C^*(G) \otimes_{\max} C^*(H)$;
- (3) $\mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v}) = \mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v})$.

Proof. (1) \Rightarrow (2). By Proposition 2.2, $\mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v}) \subseteq_{\text{coi}} C^*(G) \otimes_{\max} C^*(H)$. Suppose that $C^*(G) \otimes_{\max} C^*(H)$ is represented faithfully as a unital C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} . Using the identity maps $\iota_{\mathbf{u}}$ and $\iota_{\mathbf{v}}$ on $\mathcal{S}(\mathbf{u})$ and $\mathcal{S}(\mathbf{v})$ respectively, we define a unital completely positive map $\vartheta : \mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}) \rightarrow \mathcal{B}(\mathcal{H})$ via

$$\vartheta = \iota_{\mathbf{u}} \otimes \iota_{\mathbf{v}} : \mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}) \rightarrow \mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v}) \subseteq_{\text{coi}} C^*(G) \otimes_{\max} C^*(H).$$

Let $\Theta : C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v})) \rightarrow \mathcal{B}(\mathcal{H})$ be a completely positive extension of ϑ . The map Θ sends every contraction in $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$ of the form $u \otimes 1$, $u \in \mathbf{u}$, to the unitary element $u \otimes 1$ of $C^*(G) \otimes_{\max} C^*(H)$. Since the multiplicative domain of a unital completely positive map [17] contains all contractions that are sent to unitary elements, $u \otimes 1$ in $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$ is in the multiplicative domain of Θ for every $u \in \mathbf{u}$. Similarly, $1 \otimes v$ is in the multiplicative domain of Θ for every $v \in \mathbf{v}$. Because the multiplicative domain is an algebra, hypothesis (1) and Lemma 2.4 imply that $u \otimes v = (u \otimes 1)(1 \otimes v)$ is also in the multiplicative domain of Θ , for every $u \in \mathbf{u}$ and $v \in \mathbf{v}$. Because $\mathcal{S}(\mathbf{u}) \otimes \mathcal{S}(\mathbf{v})$ generates $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$, the map Θ is a $*$ -homomorphism of $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$ onto $C^*(G) \otimes_{\max} C^*(H)$.

Conversely, represent $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$ faithfully as a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} . Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ denote the C^* -subalgebra generated by the elements of the form $u \otimes 1$, $u \in \mathbf{u}$, and let \mathcal{B} be the C^* -subalgebra generated by elements of the form $1 \otimes v$, $v \in \mathbf{v}$. Hypothesis (1) and Lemma 2.4 imply that $ab = ba$ whenever $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Consider the map $\rho_1 : \mathcal{S}(\mathbf{u}) \rightarrow \mathcal{B}(\mathcal{H})$ given by $\rho_1(u) = u \otimes 1$, $u \in \mathcal{S}(\mathbf{u})$, and extend it to a completely positive map on $C^*(G)$, still denoted by ρ_1 . Because $\rho_1(u)$ is unitary for every $u \in \mathbf{u}$, we have that \mathbf{u} is contained in the multiplicative domain of ρ_1 . Hence, ρ_1 is a $*$ -homomorphism and consequently its range is in \mathcal{A} . Similarly, the map $\rho_2 : \mathcal{S}(\mathbf{v}) \rightarrow \mathcal{B}(\mathcal{H})$ given by $\rho_2(v) = 1 \otimes v$, $v \in \mathbf{v}$, extends to a $*$ -homomorphism of $C^*(H)$ into \mathcal{B} . Because \mathcal{A} and \mathcal{B} commute, we obtain a $*$ -homomorphism $\rho_1 \otimes \rho_2 : C^*(G) \otimes_{\max} C^*(H) \rightarrow \mathcal{B}(\mathcal{H})$, whose range is $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$. This map is the inverse of Θ on the set of all generators and is hence $\rho_1 \otimes \rho_2 = \Theta^{-1}$. Thus, Θ is a $*$ -isomorphism between $C^*(G) \otimes_{\max} C^*(H)$ and $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$ and so (1) implies (2).

(2) \Rightarrow (3). Assumption (2) and the definition of the tensor product ess yield

$$\mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v}) \subseteq_{\text{coi}} C^*(G) \otimes_{\max} C^*(H) = C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v})).$$

On the other hand, we trivially have

$$\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}) \subseteq_{\text{coi}} C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v})).$$

It follows that $\mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v}) = \mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v})$.

(3) \Rightarrow (1). Let $\iota_{\mathbf{u}} : \mathcal{S}(\mathbf{u}) \rightarrow C^*(G)$ and $\iota_{\mathbf{v}} : \mathcal{S}(\mathbf{v}) \rightarrow C^*(H)$ be the inclusion maps, $\phi = \iota_{\mathbf{u}} \otimes \iota_{\mathbf{v}}$ and let $\mathcal{R} = \phi(\mathcal{S}(\mathbf{u}) \otimes \mathcal{S}(\mathbf{v})) \subseteq C^*(G) \otimes C^*(H)$. By the definition of ess and Proposition 2.2, ϕ is a complete order isomorphism of the operator system $\mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v})$ and the operator subsystem $\mathcal{R} \subseteq C^*(G) \otimes_{\max} C^*(H)$. Because \mathcal{R} generates $C^*(G) \otimes_{\max} C^*(H)$ as a C^* -algebra, the universal property of the C^* -envelope implies that there exists a $*$ -epimorphism $\pi : C^*(G) \otimes_{\max} C^*(H) \rightarrow C_e^*(\mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v}))$ such that $\pi(x) = \iota_e(x)$ for every $x \in \mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v})$. Because $u \otimes 1, 1 \otimes v \in \mathcal{R}$ are unitary elements in $C^*(G) \otimes_{\max} C^*(H)$, $\iota_e(u \otimes 1) = \pi(u \otimes 1)$ and $\iota_e(1 \otimes v) = \pi(1 \otimes v)$ are unitary elements of $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v}))$. By invoking the hypothesis $\mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v}) = \mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v})$, we have that $u \otimes 1$ and $1 \otimes v$ are unitary in $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_\tau \mathcal{S}(\mathbf{v}))$, for every $u \in \mathbf{u}$ and every $v \in \mathbf{v}$. \square

Corollary 2.6. *Let \mathbf{u} and \mathbf{v} be generating sets of discrete groups G and H , respectively. Then $\mathcal{S}(\mathbf{u}) \otimes_{\text{ess}} \mathcal{S}(\mathbf{v}) = \mathcal{S}(\mathbf{u}) \otimes_c \mathcal{S}(\mathbf{v})$ if and only if $u \otimes 1$ and $1 \otimes v$ are unitary elements of $C_e^*(\mathcal{S}(\mathbf{u}) \otimes_c \mathcal{S}(\mathbf{v}))$ for every $u \in \mathbf{u}$ and $v \in \mathbf{v}$.*

Several discrete groups satisfy the hypothesis of Theorem 2.5, as we shall see in subsequent sections of the paper.

3. OPERATOR SYSTEMS ON FINITELY GENERATED FREE GROUPS

In this section, we study in detail the operator systems of free groups. Note that definition of $\mathcal{S}(\mathbf{u})$ depends in general on the choice of the generating set $\mathbf{u} \subseteq G$. We first show that with free groups, we can dispense with this dependence.

Proposition 3.1. *If $\mathbf{u} = \{u_1, \dots, u_n\}$ and $\mathbf{v} = \{v_1, \dots, v_n\}$ are two sets of generators of the free group \mathbb{F}_n , then there is a complete order isomorphism $\phi : \mathcal{S}(\mathbf{u}) \rightarrow \mathcal{S}(\mathbf{v})$ such that $\phi(u_j) = v_j$, $j = 1, \dots, n$.*

Proof. By the universality of $C^*(\mathbb{F}_n)$, there is a $*$ -automorphism ϑ of $C^*(\mathbb{F}_n)$ for which $\vartheta(u_j) = v_j$, $j = 1, \dots, n$. Thus, $\phi = \vartheta|_{\mathcal{S}(\mathbf{u})}$ is a complete order isomorphism from $\mathcal{S}(\mathbf{u})$ onto $\mathcal{S}(\mathbf{v})$. \square

Henceforth, let $\mathbf{u} = \{u_1, \dots, u_n\}$ and $\mathbf{v} = \{v_1, \dots, v_m\}$ be generators for the free groups \mathbb{F}_n and \mathbb{F}_m respectively and denote by \mathcal{S}_n and \mathcal{S}_m , respectively, the operator subsystems $\mathcal{S}(\mathbf{u}) \subseteq C^*(\mathbb{F}_n)$ and $\mathcal{S}(\mathbf{v}) \subseteq C^*(\mathbb{F}_m)$. By Proposition 3.1, we may identify \mathcal{S}_n and \mathcal{S}_m in the case where $m = n$. Moreover, we write

$$\mathcal{S}_n = \text{Span} \{u_{-n}, \dots, u_{-1}, u_0, u_1, \dots, u_n\},$$

where $u_{-k} = u_k^*$, $k = 1, \dots, n$, and $u_0 = 1$.

Throughout, ι_k shall denote the inclusion of \mathcal{S}_k into $C^*(\mathbb{F}_k)$.

Lemma 3.2. $\mathcal{S}_n \otimes_c \mathcal{S}_m \subseteq_{\text{coi}} C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_m)$ and, consequently, $\mathcal{S}_n \otimes_c \mathcal{S}_m = \mathcal{S}_n \otimes_{\text{ess}} \mathcal{S}_m$.

Proof. Since $\mathcal{R} \otimes_c \mathcal{S}_n \subseteq_{\text{coi}} \mathcal{R} \otimes_c C^*(\mathbb{F}_n)$ for every $n \in \mathbb{N}$ and every operator system \mathcal{R} [6, Lemma 4.1], and because $c = \max$ if one of the tensor factors is a unital C^* -algebra [14, Theorem 6.7], we deduce that

$$\mathcal{S}_n \otimes_c \mathcal{S}_m \subseteq_{\text{coi}} \mathcal{S}_n \otimes_{\max} C^*(\mathbb{F}_m) = \mathcal{S}_n \otimes_c C^*(\mathbb{F}_m) \subseteq_{\text{coi}} C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_m).$$

Hence, $\mathcal{S}_n \otimes_c \mathcal{S}_m \subseteq_{\text{coi}} C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_m)$. The last statement follows by the definition of ess and Proposition 2.2. \square

Proposition 3.3. *Let \mathcal{R} be an operator system. Then $\mathcal{R} \otimes_{\min} \mathcal{S}_1 = \mathcal{R} \otimes_c \mathcal{S}_1$. In particular, $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1 = \mathcal{S}_1 \otimes_c \mathcal{S}_1$.*

Proof. Since $C^*(\mathbb{F}_1)$ is $*$ -isomorphic to $C(\mathbb{T})$, we have that $\mathcal{R} \otimes_{\min} \mathcal{S}_1 \subseteq_{\text{coi}} \mathcal{R} \otimes_{\min} C(\mathbb{T})$. On the other hand, [6, Lemma 4.1] implies that $\mathcal{R} \otimes_c \mathcal{S}_1 \subseteq_{\text{coi}} \mathcal{R} \otimes_c C(\mathbb{T})$. Since $C(\mathbb{T})$ is a nuclear C^* -algebra, and hence a nuclear operator system, we conclude that $\mathcal{R} \otimes_{\min} C(\mathbb{T}) = \mathcal{R} \otimes_c C(\mathbb{T})$, and thus $\mathcal{R} \otimes_{\min} \mathcal{S}_1 = \mathcal{R} \otimes_c \mathcal{S}_1$. \square

In the lemma below, the vector spaces $\mathcal{M}_p(\mathcal{R} \otimes \mathcal{T})$ and $\mathcal{M}_p(\mathcal{R}) \otimes \mathcal{T}$ are canonically identified, and ϕ_m denotes the map defined in Example 1.3.

Lemma 3.4. *If $Y = \sum_{\ell=-m}^m Y_\ell \otimes u_\ell$ is strictly positive in $\mathcal{M}_p(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$, then there exist $A_{ij} \in \mathcal{M}_p(C^*(\mathbb{F}_n))$, where $i, j \in \{1, \dots, m+1\}$ and $|i-j| \leq 1$, such that:*

- (i) *each A_{ii} is strictly positive for each i and $\frac{1}{m} \sum_{i=1}^{m+1} A_{ii} = Y_0$;*
- (ii) *$A_{j,j+1} = Y_j$ and $A_{j+1,j} = Y_{-j}$, for all $j = 1, \dots, m$;*
- (iii) $X = \sum_{|i-j| \leq 1} A_{ij} \otimes E_{ij}$ *is strictly positive in $\mathcal{M}_p(C^*(\mathbb{F}_n)) \otimes_{\max} \mathcal{T}_{m+1}$;*

$$(iv) \ Y = (\text{id}_{C^*(\mathbb{F}_n)} \otimes \phi_m)^{(p)}(X).$$

Proof. Recall that $c = \max$ if one of the tensor factors is a C^* -algebra [14, Theorem 6.7]. Thus,

$$\mathcal{S}_n \otimes_{\max} \mathcal{S}_m \subseteq^+ \mathcal{S}_n \otimes_c \mathcal{S}_m \subseteq_{\text{coi}} C^*(\mathbb{F}_n) \otimes_c \mathcal{S}_m = C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{S}_m,$$

and so the strictly positive element $Y \in \mathcal{M}_p(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$ is also strictly positive in $\mathcal{M}_p(C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{S}_m)$. Let $\vartheta_{n,m} : C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{T}_{m+1} \rightarrow C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{S}_m$ be given by $\vartheta_{n,m} = \text{id}_{C^*(\mathbb{F}_n)} \otimes \phi_m$. By [7, Proposition 1.6, Theorem 4.2], $\vartheta_{n,m}$ is a complete quotient map of $C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{T}_{m+1}$ onto $C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{S}_m$. By Proposition 1.5, there is a strictly positive element $X \in \mathcal{M}_p(C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{T}_{m+1})$ such that $\vartheta_{n,m}^{(p)}(X) = Y$. Express X as

$$X = \sum_{|i-j| \leq 1} A_{ij} \otimes E_{ij},$$

for some $A_{ij} \in \mathcal{M}_p(C^*(\mathbb{F}_n))$. Thus,

$$\begin{aligned} Y = \vartheta_{n,m}^{(p)}(X) &= \sum_{|i-j| \leq 1} A_{ij} \otimes \phi_m(E_{ij}) \\ &= \frac{1}{m} \sum_{i=1}^{m+1} A_{ii} \otimes u_0 + \frac{1}{m} \sum_{j=1}^m [(A_{j+1,j} \otimes u_{-j}) + (A_{j,j+1} \otimes u_j)], \end{aligned}$$

which implies that $\frac{1}{m} \sum_{i=1}^{m+1} A_{ii} = Y_0$, $A_{j,j+1} = Y_j$ and $A_{j+1,j} = Y_{-j}$, for all $j = 1, \dots, m$. Since each A_{ii} is attained from X by a formal matrix product QXQ for a projection $Q \in \mathcal{M}_p(\mathbb{C})$, each A_{ii} is strictly positive. \square

The image of \mathcal{S}_1 under the identification of $C^*(\mathbb{F}_1)$ with $C(\mathbb{T})$ is $\text{span}\{1, z, \bar{z}\}$, where z is the identity function. By abuse of notation, we will use the symbol z to denote the variable in \mathbb{T} , which will cause no confusion as it will be clear from the context whether we refer to the identity function or the corresponding variable. If \mathcal{T} is any vector space then $\mathcal{T} \otimes \mathcal{S}_1$ will be identified in a natural way with the space of all functions from \mathbb{T} into \mathcal{T} of the form $\alpha_0 + \alpha_1 z + \alpha_2 \bar{z}$, $\alpha_0, \alpha_1, \alpha_2 \in \mathcal{T}$.

Corollary 3.5. *For every strictly positive element $Y \in \mathcal{M}_2(\mathcal{S}_1 \otimes_{\max} \mathcal{S}_1)$, where*

$$Y = Y_0 \otimes 1 + Y_1 \otimes z + Y_1^* \otimes \bar{z}$$

for some $Y_0, Y_1 \in \mathcal{M}_2(\mathcal{S}_1)$, there exist strictly positive elements $A, B \in \mathcal{M}_2(C(\mathbb{T}))$ such that $A + B = Y_0$ and

$$X = \begin{bmatrix} A & Y_1 \\ Y_1^* & B \end{bmatrix}$$

is strictly positive in $\mathcal{M}_4(C(\mathbb{T}))$.

Proof. By Lemma 3.4, there exists such an X in $\mathcal{M}_2(C^*(\mathbb{F}_1) \otimes_{\max} \mathcal{T}_2) = \mathcal{M}_2(C(\mathbb{T}) \otimes_{\max} \mathcal{T}_2)$. But, by the nuclearity of $C(\mathbb{T})$, we have that

$$\mathcal{M}_2(C(\mathbb{T}) \otimes_{\max} \mathcal{T}_2) = \mathcal{M}_2(C(\mathbb{T}) \otimes_{\min} \mathcal{T}_2) \subseteq_{\text{coi}} \mathcal{M}_2(C(\mathbb{T}) \otimes_{\min} \mathcal{M}_2) = \mathcal{M}_4(C(\mathbb{T})),$$

which yields the result. \square

In Corollary 3.5 above, the diagonal elements of the matrix $\begin{bmatrix} A & Y_1 \\ Y_1^* & B \end{bmatrix}$ belong to $\mathcal{M}_2(C(\mathbb{T}))$. In fact, it is possible for these diagonal entries to be chosen from $\mathcal{M}_2(\mathcal{S}_1)$, as we now demonstrate.

Lemma 3.6. *Let \mathcal{T} be an operator system. If, for some $t_0, t_1 \in \mathcal{T}$, the element*

$$(5) \quad y = 1 \otimes t_0 + z \otimes t_1 + \bar{z} \otimes t_1^* \in \mathcal{S}_1 \otimes_{\max} \mathcal{T},$$

is strictly positive, then there are $t_0^1, t_0^2 \in \mathcal{T}_+$ such that $t_0^1 + t_0^2 = t_0$ and

$$(6) \quad Y = \begin{bmatrix} t_0^1 & t_1 \\ t_1^* & t_0^2 \end{bmatrix} \in \mathcal{M}_2(\mathcal{T})_+.$$

Conversely, if $Y \in \mathcal{M}_2(\mathcal{T})_+$ is the matrix (6), then the element y in (5), where $t_0 = t_0^1 + t_0^2$, is positive in $\mathcal{S}_1 \otimes_{\max} \mathcal{T}$.

Proof. Let $1 \otimes t_0 + z \otimes t_1 + \bar{z} \otimes t_1^*$ be strictly positive in $\mathcal{S}_1 \otimes_{\max} \mathcal{T}$. By Lemma 1.7, there exist $n \in \mathbb{N}$, $F = [f_{ij}]_{i,j} \in \mathcal{M}_n(\mathcal{S}_1)_+$, and $T = [t_{ij}]_{i,j} \in \mathcal{M}_n(\mathcal{T})_+$ such that

$$1 \otimes t_0 + z \otimes t_1 + \bar{z} \otimes t_1^* = \sum_{i,j=1}^n f_{ij} \otimes t_{ij}.$$

Write $f_{ij} = \alpha_{ij} + \beta_{ij}z + \overline{\beta_{ij}z}$, for some $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$. Thus,

$$1 \otimes t_0 + z \otimes t_1 + \bar{z} \otimes t_1^* = \sum_{i,j=1}^n (\alpha_{ij} + \beta_{ij}z + \overline{\beta_{ij}z}) \otimes t_{ij},$$

which implies that

$$1 \otimes t_0 = \sum_{i,j=1}^n \alpha_{ij} \otimes t_{ij} = 1 \otimes \sum_{i,j=1}^n \alpha_{ij} t_{ij}$$

and

$$z \otimes t_1 = \sum_{i,j=1}^n \beta_{ij} z \otimes t_{ij} = z \otimes \sum_{i,j=1}^n \beta_{ij} t_{ij}.$$

Hence,

$$t_0 = \sum_{i,j=1}^n \alpha_{ij} t_{ij} \quad \text{and} \quad t_1 = \sum_{i,j=1}^n \beta_{ij} t_{ij}.$$

Let $A = [\alpha_{ij}]_{i,j}$ and $B = [\beta_{ij}]_{i,j} \in \mathcal{M}_n(\mathbb{C})$. Since $F(z) \in \mathcal{M}_n(\mathbb{C})_+$ for every $z \in \mathbb{T}$, the matrix A is positive while $\bar{B} = B^*$. For every $\delta > 0$, the matrix $A_\delta := A + \delta I$ is positive and invertible and hence

$$A_\delta^{-1/2}(F(z) + \delta I)A_\delta^{-1/2} = I + A_\delta^{-1/2}BA_\delta^{-1/2}z + A_\delta^{-1/2}B^*A_\delta^{-1/2}\bar{z}$$

is positive for every $z \in \mathbb{T}$. It follows that $\frac{1}{2}$ is an upper bound on the numerical radius of $A_\delta^{-1/2}BA_\delta^{-1/2}$ and hence, by Ando's theorem [1], there exist matrices

$$Q_{1,\delta}, Q_{2,\delta} \in \mathcal{M}_n(\mathbb{C})_+ \text{ such that } Q_{1,\delta} + Q_{2,\delta} = I \text{ and } \begin{bmatrix} Q_{1,\delta} & A_\delta^{-1/2}BA_\delta^{-1/2} \\ A_\delta^{-1/2}B^*A_\delta^{-1/2} & Q_{2,\delta} \end{bmatrix}$$

is positive. Therefore, if $A_{j,\delta} = A_\delta^{1/2}Q_{j,\delta}A_\delta^{1/2}$, $j = 1, 2$, then $A_{1,\delta} + A_{2,\delta} = A_\delta$ and $\begin{bmatrix} A_{1,\delta} & B \\ B^* & A_{2,\delta} \end{bmatrix}$ is positive. Since the matrices $A_{1,\delta}, A_{2,\delta}$ are uniformly bounded for $\delta \in (0, 1)$, by passing to a limit point we obtain positive matrices A_1 and A_2 such that $A_1 + A_2 = A$ and $\begin{bmatrix} A_1 & B \\ B^* & A_2 \end{bmatrix}$ is positive. By Lemma 1.6, $\begin{bmatrix} A_1 \circ T & B \circ T \\ B^* \circ T & A_2 \circ T \end{bmatrix}$ is positive in $\mathcal{M}_{2n}(\mathcal{T})$.

For any operator system \mathcal{R} , the map $\sigma_n : \mathcal{M}_n(\mathcal{R}) \rightarrow \mathcal{R}$ defined by $\sigma_n([r_{ij}]_{i,j}) = \sum_{i,j} r_{ij}$ is completely positive. Hence, with $(\alpha_{ij}^{[k]})_{i,j} = A_k$ and $t_0^k = \sum_{i,j} \alpha_{ij}^{[k]} t_{ij}$, for $k = 1, 2$, we have that

$$\begin{bmatrix} t_0^1 & t_1 \\ t_1^* & t_0^2 \end{bmatrix} = \begin{bmatrix} \sum_{i,j} \alpha_{ij}^{[1]} t_{ij} & \sum_{i,j} \beta_{ij} t_{ij} \\ \sum_{i,j} \bar{\beta}_{ij} t_{ij}^* & \sum_{i,j} \alpha_{ij}^{[2]} t_{ij} \end{bmatrix} = \sigma_n^{(2)} \left(\begin{bmatrix} A_1 \circ T & B \circ T \\ B^* \circ T & A_2 \circ T \end{bmatrix} \right)$$

is positive in $\mathcal{M}_2(\mathcal{T})$.

Conversely, assume that $Y = \begin{bmatrix} t_0^1 & t_1 \\ t_1^* & t_0^2 \end{bmatrix} \in \mathcal{M}_2(\mathcal{T})_+$, and set $t_0 = t_0^1 + t_0^2$. Because matrix $Z = \begin{bmatrix} 1 & z \\ \bar{z} & 1 \end{bmatrix}$ is positive in $\mathcal{M}_2(\mathcal{S}_1)$, the element $y = 1 \otimes (t_0^1 + t_0^2) + z \otimes t_1 + \bar{z} \otimes t_1^*$ is positive in $\mathcal{S}_1 \otimes_{\max} \mathcal{T}$ [14, Lemma 5.2]. \square

We now arrive at one of the main results of this paper. Recall that $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1 = \mathcal{S}_1 \otimes_c \mathcal{S}_1$ (Proposition 3.3). The following theorem, therefore, shows that $\mathcal{S}_1 \otimes_c \mathcal{S}_1 \neq \mathcal{S}_1 \otimes_{\max} \mathcal{S}_1$.

Theorem 3.7. *If $\iota = \text{id}_{\mathcal{S}_1} \otimes \text{id}_{\mathcal{S}_1}$, then $\iota : \mathcal{S}_1 \otimes_{\min} \mathcal{S}_1 \rightarrow \mathcal{S}_1 \otimes_{\max} \mathcal{S}_1$ is a positive linear map that is not 2-positive.*

Proof. To prove that ι is a positive map, note that $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1 \subseteq_{\text{coi}} C(\mathbb{T}) \otimes_{\min} C(\mathbb{T}) = C(\mathbb{T} \times \mathbb{T})$. Let z and w denote the images of the two generators, viewed as functions on the torus \mathbb{T}^2 . We may write a typical element of $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1$ as $f = b_{-1} \otimes \bar{z} + b_0 \otimes 1 + b_1 \otimes z$, where $b_i \in \text{span}\{\bar{w}, 1, w\}$. Such an element f is positive as a function on the torus if and only if $b_{-1} = \bar{b}_1$ and $b_0 \geq 2|b_1|$. But these conditions guarantee that the matrix

$$\begin{bmatrix} b_0/2 & b_1 \\ b_{-1} & b_0/2 \end{bmatrix},$$

is in $\mathcal{M}_2(\mathcal{S}_1)_+$. We also have that

$$\begin{bmatrix} 1 & z \\ \bar{z} & 1 \end{bmatrix}$$

belongs to $\mathcal{M}_2(\mathcal{S}_1)_+$. Thus,

$$f = b_0/2 \otimes 1 + b_1 \otimes z + b_{-1} \otimes \bar{z} + b_0/2 \otimes 1$$

is in $(\mathcal{S}_1 \otimes_{\max} \mathcal{S}_1)_+$ [14, Lemma 5.2]. That is, every element of $(\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1)_+$ is also in $(\mathcal{S}_1 \otimes_{\max} \mathcal{S}_1)_+$, which proves the positivity of the map ι .

To show that ι is not 2-positive, consider the element $h \in \mathcal{M}_2(\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1)$ which, considered as an $\mathcal{M}_2(\mathbb{C})$ -valued function on the variables $z, w \in \mathbb{T}$, is given by

$$(7) \quad h(z, w) = \begin{bmatrix} 3 + 2\Re(zw) & 2\bar{z}w \\ 2z\bar{w} & 3 - 2\Re(zw) \end{bmatrix}.$$

Because $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1 \subseteq_{\text{coi}} C(\mathbb{T}) \otimes_{\min} C(\mathbb{T})$, h is positive in $\mathcal{M}_2(\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1)$ if and only if $h(z, w) \in \mathcal{M}_2(\mathbb{C})_+$ for every $z, w \in \mathbb{T}$. The matrix

$$k(z, w) = \begin{bmatrix} 2\Re(zw) & 2\bar{z}w \\ 2z\bar{w} & -2\Re(zw) \end{bmatrix},$$

has characteristic polynomial $\lambda^2 - (2\Re(zw))^2 - 4$. Thus, the eigenvalues of the hermitian matrix $h(z, w) = 3I + k(z, w)$ are uniformly bounded below by $3 - 2\sqrt{2} > 0$, which shows that h is strictly positive in $\mathcal{M}_2(\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1)$. We now show that h is not a (strictly) positive element of $\mathcal{M}_2(\mathcal{S}_1 \otimes_{\max} \mathcal{S}_1)$.

Suppose, contrary to what we aim to prove, that h is in fact strictly positive in $\mathcal{M}_2(\mathcal{S}_1 \otimes_{\max} \mathcal{S}_1)$. By the identification of $\mathcal{M}_2(\mathcal{S}_1 \otimes \mathcal{S}_1)$ with $\mathcal{S}_1 \otimes \mathcal{M}_2(\mathcal{S}_1)$, rewrite h in tensor notation:

$$(8) \quad h = 1 \otimes \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + z \otimes \begin{bmatrix} w & 0 \\ 2\bar{w} & -w \end{bmatrix} + \bar{z} \otimes \begin{bmatrix} \bar{w} & 2w \\ 0 & -\bar{w} \end{bmatrix}.$$

Thus, because h is strictly positive, Lemma 3.6 implies that there is a strictly positive

$$X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \in \mathcal{M}_4(\mathcal{S}_1)$$

such that $A + B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $C(w) = \begin{bmatrix} w & 0 \\ 2\bar{w} & -w \end{bmatrix}$ in $\mathcal{M}_2(\mathcal{S}_1)$. Let $R = U^* X U$, where $U = \text{Diag}(w^2, 1, w, \bar{w}) \in \mathcal{M}_4(C(\mathbb{T}))$. Thus, as a function $R : \mathbb{T} \rightarrow \mathcal{M}_4(\mathbb{C})$, R is given by

$$R(w) = \left[\begin{array}{cc|cc} f_{11}(w) & \bar{w}^2 f_{12}(w) & 1 & 0 \\ w^2 f_{21}(w) & f_{22}(w) & 2 & -1 \\ \hline 1 & 2 & 3 - f_{11}(w) & -\bar{w}^2 f_{12}(w) \\ 0 & -1 & -w^2 f_{21}(w) & 3 - f_{22}(w) \end{array} \right],$$

where $f_{ij} \in \mathcal{S}_1$ and is of the form $f_{ij}(w) = \alpha_{ij} + \beta_{ij}w + \gamma_{ij}\bar{w}$, for $i, j \in \{1, 2\}$. Let $g_{21}(w) = w^2 f_{21}(w) = \alpha_{21}w^2 + \beta_{21}w^3 + \gamma_{21}w$. Because $R(w)$ is hermitian, $\bar{w}^2 f_{12}(w) = \overline{g_{21}(w)}$.

Let $\zeta = e^{i\frac{2\pi}{5}}$. Then ζ, ζ^2 , and ζ^3 all generate the same cyclic subgroup C of \mathbb{T} , namely $C \cong \mathbb{Z}_5$. Because $\zeta + \zeta^2 + \dots + \zeta^5 = 0$, we deduce that

$$\frac{1}{5} \sum_{k=1}^5 f_{jj}(\zeta^k) = \alpha_{jj} \quad \text{and} \quad \frac{1}{5} \sum_{k=1}^5 g_{21}(\zeta^k) = 0.$$

Thus, in $\mathcal{M}_4(\mathbb{C})$,

$$0 \leq \frac{1}{5} \sum_{k=1}^5 R(\zeta^k) = \left[\begin{array}{cc|cc} \alpha_{11} & 0 & 1 & 0 \\ 0 & \alpha_{22} & 2 & -1 \\ \hline 1 & 2 & 3 - \alpha_{11} & 0 \\ 0 & -1 & 0 & 3 - \alpha_{22} \end{array} \right].$$

Let $b \in \mathcal{M}_4(\mathbb{C})$ denote the positive matrix above, which in 2×2 block form we write as $b = \begin{bmatrix} \alpha_{11} & b_{12} \\ b_{12}^* & b_{22} \end{bmatrix}$. By the Cholesky Algorithm, there is a matrix Z for which

$b = Z^* \begin{bmatrix} \alpha_{11} & 0 \\ 0 & \tilde{b}_{22} \end{bmatrix} Z$, where

$$0 \leq \tilde{b}_{22} = b_{22} - \frac{1}{\alpha_{11}} b_{12}^* b_{12} = \begin{bmatrix} \alpha_{22} & 2 & -1 \\ 2 & 3 - \alpha_{11} - \frac{1}{\alpha_{11}} & 0 \\ -1 & 0 & 3 - \alpha_{22} \end{bmatrix}.$$

As $2 \leq \alpha_{11} + \frac{1}{\alpha_{11}}$, the matrix

$$c = \begin{bmatrix} \alpha_{22} & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 3 - \alpha_{22} \end{bmatrix}$$

is positive. Thus,

$$0 \leq \det(c) = -\alpha_{22}^2 + 7\alpha_{22} - 13.$$

The polynomial $p(x) = x^2 - 7x + 13$ has no real roots and $p(0) = 13 > 0$. Thus, $p(x) > 0$ for every $x \in \mathbb{R}$ contradicting $p(\alpha_{22}) \leq 0$. Therefore, h is not a positive element of $\mathcal{M}_2(\mathcal{S}_1 \otimes_{\max} \mathcal{S}_1)$. \square

Some important applications of Theorem 3.7 are given below in Theorems 3.8 and 3.9.

Theorem 3.8. *The identity map from $\mathcal{S}_n \otimes_c \mathcal{S}_m$ to $\mathcal{S}_n \otimes_{\max} \mathcal{S}_m$ is not 2-positive for all $n, m \geq 1$.*

Proof. Since $C^*(\mathbb{F}_1)$ can be canonically identified with a C^* -subalgebra of $C^*(\mathbb{F}_n)$, the canonical map $\iota_n : \mathcal{S}_1 \rightarrow \mathcal{S}_n$ is a complete order inclusion. By the universal property of \mathcal{S}_n [13, Proposition 9.7], we have a completely positive map $\psi_n : \mathcal{S}_n \rightarrow \mathcal{S}_1$ defined by sending the generators $\{u_2, \dots, u_n, u_{-2}, \dots, u_{-n}\}$ to 0 and fixing u_1 and 1. Thus, if for any n, m we had that $\gamma : \mathcal{S}_n \otimes_c \mathcal{S}_m \rightarrow \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$ were 2-positive, then we would have that $(\psi_n \otimes \psi_m) \circ \gamma \circ (\iota_n \otimes \iota_m) : \mathcal{S}_1 \otimes_c \mathcal{S}_1 \rightarrow \mathcal{S}_1 \otimes_{\max} \mathcal{S}_1$ is 2-positive, contradicting in this way Theorem 3.7. \square

Theorem 3.9. *We have that $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m) \neq C_e^*(\mathcal{S}_n) \otimes_{\max} C_e^*(\mathcal{S}_m)$ for every $n, m \geq 1$.*

Proof. By Proposition 2.2, $C_e^*(\mathcal{S}_n) = C^*(\mathbb{F}_n)$. By Lemma 3.2, $\mathcal{S}_n \otimes_c \mathcal{S}_m \subseteq_{\text{coi}} C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_m)$; in other words, $\mathcal{S}_n \otimes_c \mathcal{S}_m = \mathcal{S}_n \otimes_{\text{ess}} \mathcal{S}_m$. The hypothesis of Theorem 2.5 is clearly satisfied for $\tau = \max$. By Theorem 3.8, $\mathcal{S}_n \otimes_c \mathcal{S}_m \neq \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$, and hence Theorem 2.5 yields $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m) \neq C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_m)$. \square

Proposition 3.10. *The map $\phi_2 \otimes_{\min} \phi_2 : \mathcal{M}_2 \otimes_{\min} \mathcal{M}_2 \rightarrow \mathcal{S}_1 \otimes_{\min} \mathcal{S}_1$ is not a complete quotient map.*

Proof. Since $\phi_2 \otimes_{\max} \phi_2 : \mathcal{M}_2 \otimes_{\max} \mathcal{M}_2 \rightarrow \mathcal{S}_1 \otimes_{\max} \mathcal{S}_1$ is completely positive and $\mathcal{M}_2 \otimes_{\min} \mathcal{M}_2 = \mathcal{M}_2 \otimes_{\max} \mathcal{M}_2$, if $\phi_2 \otimes_{\min} \phi_2$ were a complete quotient map, then we could lift every positive element in $P \in \mathcal{M}_2(\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1)_+$ to a positive element in $R \in \mathcal{M}_2(\mathcal{M}_2 \otimes_{\min} \mathcal{M}_2)_+$, which would imply that $(\phi_2 \otimes_{\max} \phi_2)(R)$ is positive in $\mathcal{S}_1 \otimes_{\max} \mathcal{S}_1$. However, this would show that the identity map from $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1$ to $\mathcal{S}_1 \otimes_{\max} \mathcal{S}_1$ is 2-positive, contradicting Theorem 3.7. \square

Let \mathbb{S}_n denote the group of permutations of $\{1, \dots, n\}$ and select $\sigma \in \mathbb{S}_n$. The linear map $\phi_\sigma : \mathcal{S}_n \rightarrow \mathcal{S}_n$ defined on the canonical basis of \mathcal{S}_n by $\phi_\sigma(u_i) = u_{\sigma(i)}$, is the restriction of a $*$ -automorphism of $C^*(G)$ and is hence a complete order isomorphism. Fix another element $\tau \in \mathbb{S}_m$. Since ϕ_σ and ϕ_τ are complete order isomorphisms, for any tensor product α that is *functorial* in the sense of [14], the canonical tensor product map $\phi_\sigma \otimes_\alpha \phi_\tau : \mathcal{S}_n \otimes_\alpha \mathcal{S}_m \rightarrow \mathcal{S}_n \otimes_\alpha \mathcal{S}_m$ is a complete order isomorphism. It was shown in [14] that \min, c and \max are all functorial. We note that we do not know whether the tensor product ess is functorial.

Proposition 3.11. *In $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$, $u_i \otimes 1$ fails to be a unitary element for every i and $1 \otimes v_j$ fails to be a unitary element for every j .*

Proof. If $\sigma \in \mathbb{S}_n$, then $\phi_\sigma \otimes \text{id}$ is a complete order isomorphism $\mathcal{S}_n \otimes_{\max} \mathcal{S}_m \rightarrow \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$, by the preceding observation. Therefore, by the universal property of C^* -envelopes, there is an epimorphism $\pi_\sigma : C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m) \rightarrow C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$ such that $u_{\sigma(i)} \otimes 1 = \pi_\sigma(u_i \otimes 1)$ for every i . Thus, if $u_i \otimes 1$ is a unitary element of $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$ for some i , then $u_\ell \otimes 1$ is unitary for every ℓ . A similar statement holds for the elements $1 \otimes v_j$. But, by Theorems 2.5 and 3.8, at least one element in

the set $\{u_i \otimes 1, 1 \otimes v_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ fails to be unitary in $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$. Hence, $u_i \otimes 1$ fails to be a unitary element of $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$ for every i or $1 \otimes v_j$ fails to be a unitary element of $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$ for every j .

Now consider the case where $n = m$ and let $\sigma : \mathcal{S}_n \otimes_{\max} \mathcal{S}_n \rightarrow \mathcal{S}_n \otimes_{\max} \mathcal{S}_n$ be the “flip” map $\sigma(x \otimes y) = y \otimes x$. This map is a complete order isomorphism and hence induces a $*$ -automorphism $\pi : C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_n) \rightarrow C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_n)$ with $\pi(u_i \otimes 1) = 1 \otimes v_i$. Hence, in this case both $u_i \otimes 1$ and $1 \otimes v_j$ must be non-unitary.

Now let $n < m$ and note that, as in the proof of Theorem 3.8, the linear map $\iota : \mathcal{S}_n \rightarrow \mathcal{S}_m$, given by $\iota(u_j) = v_j$, is a unital complete order isomorphism onto its range and the linear map $\phi : \mathcal{S}_m \rightarrow \mathcal{S}_n$ defined by $\phi(v_j) = u_j, 1 \leq j \leq n$ and $\phi(v_j) = 0, n < j \leq m$, is a unital completely positive map that is a left inverse of ι . By the functoriality of \max , the map $\text{id}_n \otimes \iota : \mathcal{S}_n \otimes_{\max} \mathcal{S}_n \rightarrow \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$ is completely positive with completely positive left inverse $\text{id}_n \otimes \phi$. Thus, the image $\mathcal{R} = \text{span}\{1 \otimes 1, u_i \otimes 1, 1 \otimes v_j, u_i \otimes v_j : 1 \leq i, j \leq n\}$ of $\text{id}_n \otimes \iota$ inside $\mathcal{S}_n \otimes_{\max} \mathcal{S}_m$ is an operator system that is completely order isomorphic to $\mathcal{S}_n \otimes_{\max} \mathcal{S}_n$.

Let $C^*(\mathcal{R})$ be the C^* -subalgebra of $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$ generated by \mathcal{R} . By the universal property of the C^* -envelope, there exists a surjective $*$ -homomorphism $\pi : C^*(\mathcal{R}) \rightarrow C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_n)$ which fixes $u_i \otimes 1$ and $1 \otimes v_j$ for $1 \leq i, j \leq n$. If $u_i \otimes 1$ or $1 \otimes v_j$ were unitary in $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_m)$ for some $i, j = 1, \dots, n$, then its image would be unitary in $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_n)$. Hence, $u_i \otimes 1$ and $1 \otimes v_j$ must be non-unitary for all $i, j = 1, \dots, n$. Permuting the generators of \mathcal{S}_m shows that $1 \otimes v_j$ fails to be unitary in $C_e^*(\mathcal{S}_n \otimes_{\max} \mathcal{S}_n)$ for all $j = 1, \dots, m$. \square

4. THE UNIVERSAL OPERATOR SYSTEMS \mathcal{S}_1 AND \mathcal{S}_1^d AND THEIR TENSOR PRODUCTS

The operator systems \mathcal{S}_1 and \mathcal{S}_1^d represent certain universal objects in operator theory, which we explain in Theorems 4.1 and 4.3 below. The dual system \mathcal{S}_1^d admits a concrete representation as a matrix operator system: namely, \mathcal{S}_1^d is (completely order isomorphic to) the operator subsystem $\text{Span}\{I, E_{12}, E_{21}\}$ of \mathcal{M}_2 [7, Theorem 4.4].

The first result in this direction is certainly well-known, but we state and prove it for completeness.

Proposition 4.1. *The operator system \mathcal{S}_1 is the universal operator system of a contraction. That is, if \mathcal{H} is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ is a contraction, then there is a unital completely positive map $\phi : \mathcal{S}_1 \rightarrow \mathcal{B}(\mathcal{H})$ with $\phi(u_1) = T$. Conversely, if $\phi : \mathcal{S}_1 \rightarrow \mathcal{B}(\mathcal{H})$ is any unital completely positive map, then $\phi(u_1)$ is a contraction.*

Proof. If $T \in \mathcal{B}(\mathcal{H})$ is a contraction, then T has a unitary dilation $U \in \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. As $C^*(\mathbb{F}_1) \equiv C(\mathbb{T})$ is the universal C^* -algebra generated by a unitary, there is a unital $*$ -homomorphism $\pi : C(\mathbb{T}) \rightarrow \mathcal{B}(\mathcal{K})$ such that $\pi(u_1) = U$. Let $\phi : \mathcal{S}_1 \rightarrow \mathcal{B}(\mathcal{H})$ be the restriction of π to \mathcal{S}_1 followed by the compression of operators in $\mathcal{B}(\mathcal{K})$ to \mathcal{H} . Then ϕ is a unital completely positive map with $\phi(u_1) = T$. The converse follows from the fact that unital completely positive maps are (completely) contractive. \square

We now consider a dual result.

Definition 4.2. The numerical radius of an operator $T \in \mathcal{B}(\mathcal{H})$ is the quantity $w(T)$ defined by

$$w(T) = \sup\{|\langle T\xi, \xi \rangle| : \xi \in \mathcal{H}, \|\xi\| = 1\}.$$

Proposition 4.3. The operator system \mathcal{S}_1^d is the universal operator system of an operator with numerical radius at most $1/2$. That is, if \mathcal{H} is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ is such that $w(T) \leq 1/2$, then there exists a unital completely positive map $\phi : \mathcal{S}_1^d \rightarrow \mathcal{B}(\mathcal{H})$ with $\phi(E_{12}) = T$. Conversely, if $\phi : \mathcal{S}_1^d \rightarrow \mathcal{B}(\mathcal{H})$ is a unital completely positive map, then $w(\phi(E_{12})) \leq 1/2$.

Proof. Given a unital completely positive map $\phi : \mathcal{S}_1^d \rightarrow \mathcal{B}(\mathcal{H})$, use Arveson's extension theorem to extend this map to a unital completely positive map $\phi : \mathcal{M}_2 \rightarrow \mathcal{B}(\mathcal{H})$. Let $T = \phi(E_{12})$ and set $A = \phi(E_{11})$, $B = \phi(E_{22})$, so that $A + B = \phi(E_{11} + E_{22}) = I_{\mathcal{H}}$. Since

$$X = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

is positive, we have that

$$(9) \quad \phi^{(2)}(X) = \begin{bmatrix} A & T \\ T^* & B \end{bmatrix}$$

is positive. By Ando's theorem [1], $w(T) \leq 1/2$.

Conversely, if $w(T) \leq 1/2$, then again by Ando's theorem there exist $A, B \in \mathcal{B}(\mathcal{H})_+$ with $A + B = I_{\mathcal{H}}$ such that the operator matrix (9) is positive. The linear map $\phi : \mathcal{M}_2 \rightarrow \mathcal{B}(\mathcal{H})$ given by $\phi(E_{12}) = T = \phi(E_{21})^*$, $\phi(E_{11}) = A$ and $\phi(E_{22}) = B$ is completely positive by Choi's criterion (see [17, Theorem 3.14]), and the proof is complete. \square

An alternate proof of Proposition 4.3 can be achieved by appealing to Arveson's nilpotent dilation theorem [2, Theorem 1.3.1] rather than Ando's theorem [1].

Theorem 4.4. The following tensor product relations involving \mathcal{S}_1 and \mathcal{S}_1^d hold:

- (1) $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1 = \mathcal{S}_1 \otimes_c \mathcal{S}_1$;
- (2) $\mathcal{S}_1 \otimes_c \mathcal{S}_1 \neq \mathcal{S}_1 \otimes_{\max} \mathcal{S}_1$;
- (3) $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1^d = \mathcal{S}_1 \otimes_c \mathcal{S}_1^d$;
- (4) $\mathcal{S}_1 \otimes_c \mathcal{S}_1^d \neq \mathcal{S}_1 \otimes_{\max} \mathcal{S}_1^d$;
- (5) $\mathcal{S}_1^d \otimes_{\min} \mathcal{S}_1^d = \mathcal{S}_1^d \otimes_c \mathcal{S}_1^d$;
- (6) $\mathcal{S}_1^d \otimes_c \mathcal{S}_1^d \neq \mathcal{S}_1^d \otimes_{\max} \mathcal{S}_1^d$.

Proof. Statements (1) and (3) follow from Proposition 3.3, while (2) follows from Theorem 3.7.

To prove (4), by [11, Proposition 4.12], we need only prove that \mathcal{S}_1^d is not completely order isomorphic to a C^* -algebra. Because every finite-dimensional C^* -algebra is necessarily injective, we need only note that \mathcal{S}_1^d is not an injective operator system. That \mathcal{S}_1^d fails to be injective is a consequence of [4, Corollary 7.2].

For statement (5), let $\phi : \mathcal{S}_1^d \rightarrow \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{S}_1^d \rightarrow \mathcal{B}(\mathcal{H})$ be unital completely positive maps with commuting ranges and let $T = \phi(E_{12})$, $R = \psi(E_{12})$. Then $w(T) \leq 1/2$, $w(R) \leq 1/2$, $RT = TR$ and $R^*T = TR^*$. If we fix real numbers $0 < r, t < 1$, then $w(tT) < 1/2$ and $w(rR) < 1/2$. Let $C^*(T)$ and $W^*(T)$ denote the unital C^* -algebra and the von Neumann algebra generated by T , respectively.

By [6, Theorem 1.1], there exist $A_t, B_t \in C^*(T)_+$ such that $A_t + B_t = I$ and the matrix

$$\begin{bmatrix} A_t & tT \\ tT^* & B_t \end{bmatrix}$$

is positive. Clearly, A_t and B_t commute with R and R^* . Now by taking limit points in the strong operator topology we obtain operators A and B in $W^*(T)$ such that $A + B = I$ and

$$\begin{bmatrix} A & T \\ T^* & B \end{bmatrix}$$

is positive. Similarly, we obtain positive operators C and D in $W^*(R)$ such that $C + D = I$ and

$$\begin{bmatrix} C & R \\ R^* & D \end{bmatrix}$$

is positive.

By Choi's criterion [17, Theorem 3.14], the maps $\tilde{\phi} : \mathcal{M}_2 \rightarrow \mathcal{B}(\mathcal{H})$ and $\tilde{\psi} : \mathcal{M}_2 \rightarrow \mathcal{B}(\mathcal{H})$ defined by $\tilde{\phi}(E_{11}) = A$, $\tilde{\phi}(E_{12}) = \tilde{\phi}(E_{21})^* = T$, $\tilde{\phi}(E_{22}) = B$ and $\tilde{\psi}(E_{11}) = C$, $\tilde{\psi}(E_{12}) = \tilde{\psi}(E_{21})^* = R$, $\tilde{\psi}(E_{22}) = D$ are completely positive extensions of ϕ and ψ , respectively, that have commuting ranges. We thus obtain a unital completely positive map $\tilde{\phi} \otimes \tilde{\psi} : \mathcal{M}_2 \otimes_c \mathcal{M}_2 = \mathcal{M}_2 \otimes_{\min} \mathcal{M}_2 \rightarrow \mathcal{B}(\mathcal{H})$. Its restriction to $\mathcal{S}_1^d \otimes_{\min} \mathcal{S}_1^d \subseteq_{\text{coi}} \mathcal{M}_2 \otimes_{\min} \mathcal{M}_2$ is clearly completely positive.

Since every pair of unital completely positive maps on \mathcal{S}_1^d with commuting ranges gives rise to a unital completely positive map on $\mathcal{S}_1^d \otimes_{\min} \mathcal{S}_1^d$, we see that $\mathcal{M}_n(\mathcal{S}_1^d \otimes_{\min} \mathcal{S}_1^d)_+ = \mathcal{M}_n(\mathcal{S}_1^d \otimes_c \mathcal{S}_1^d)_+$ for all n . Hence, $\mathcal{S}_1^d \otimes_{\min} \mathcal{S}_1^d = \mathcal{S}_1 \otimes_c \mathcal{S}_1^d$.

To prove the final assertion, we use [7, Proposition 1.9]: $(\mathcal{S} \otimes_{\min} \mathcal{T})^d = \mathcal{S}^d \otimes_{\max} \mathcal{T}^d$ and $(\mathcal{S} \otimes_{\max} \mathcal{T})^d = \mathcal{S}^d \otimes_{\min} \mathcal{T}^d$ for all finite-dimensional operator systems \mathcal{S} and \mathcal{T} . Because $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1 \neq \mathcal{S}_1 \otimes_{\max} \mathcal{S}_1$, we conclude that $\mathcal{S}_1^d \otimes_{\min} \mathcal{S}_1^d \neq \mathcal{S}_1^d \otimes_{\max} \mathcal{S}_1^d$. Hence, this inequality and $\mathcal{S}_1^d \otimes_{\min} \mathcal{S}_1^d = \mathcal{S}_1^d \otimes_c \mathcal{S}_1^d$ imply that $\mathcal{S}_1^d \otimes_c \mathcal{S}_1^d \neq \mathcal{S}_1^d \otimes_{\max} \mathcal{S}_1^d$. \square

Recall that two operators R and S are $*$ -commuting if $RS = SR$ and $RS^* = S^*R$.

Corollary 4.5. *The operator systems $\mathcal{S}_1 \otimes_{\min} \mathcal{S}_1$ and $\mathcal{S}_1^d \otimes_{\min} \mathcal{S}_1^d$ are universal operator systems, respectively, for pairs of $*$ -commuting contractions and pairs of $*$ -commuting operators of numerical radius at most $\frac{1}{2}$.*

5. TSIRELSON'S NON-COMMUTATIVE n -CUBES

Tsirelson [19, 20] studied certain operator systems that can be best thought of as the non-commutative analogues of n -dimensional cubes. To describe his ideas, we introduce an $(n+1)$ -dimensional operator system that we shall call the *operator system of the non-commuting n -cube*, and denote it by $NC(n)$.

Definition 5.1. *Let $\mathcal{G} = \{h_1, \dots, h_n\}$, let $\mathcal{R} = \{h_j^* = h_j, \|h_j\| \leq 1, 1 \leq j \leq n\}$ be a set of relations in the set \mathcal{G} , and let $C^*(\mathcal{G}|\mathcal{R})$ denote the universal unital C^* -algebra generated by \mathcal{G} subject to \mathcal{R} . The operator system*

$$NC(n) = \text{span}\{1, h_1, \dots, h_n\} \subset C^*(\mathcal{G}|\mathcal{R}).$$

is called the operator system of the non-commuting n -cube.

By the universal property of universal C^* -algebras we deduce immediately that the spectrum of each h_i in $C^*(\mathcal{G}|\mathcal{R})$ is $[-1, 1]$ and that for any n hermitian contractions A_1, \dots, A_n acting on any Hilbert space \mathcal{H} there is a unital completely positive map $\phi : NC(n) \rightarrow \mathcal{B}(\mathcal{H})$ with $\phi(h_i) = A_i$, $i = 1, \dots, n$.

If we had demanded that in addition the A_i 's pairwise commute, then for our universal operator system we could have taken the span of the constant function and the coordinate functions $x_i, 1 \leq i \leq n$, inside the C^* -algebra $C([-1, 1]^n)$ of continuous functions on the n -dimensional cube. This motivates the following definition.

Definition 5.2. *The operator system of the commutative n -cube is the operator subsystem $C(n) \subset C([-1, 1]^n)$ given by*

$$C(n) = \text{span}\{1, x_1, \dots, x_n\},$$

where x_i is the i th coordinate function on $[-1, 1]^n$.

The fact that $C(n)$ is the “universal” operator system for n commuting selfadjoint contractions is shown in the next proposition.

Proposition 5.3. *If \mathcal{H} is a Hilbert space, $A_1, \dots, A_n \in B(H)$ are any pairwise commuting operators satisfying $-I \leq A_i \leq I$, then there is a unital completely positive map $\Phi : C(n) \rightarrow \mathcal{B}(\mathcal{H})$ with $\Phi(x_i) = A_i, i = 1, \dots, n$.*

Proof. By the nuclearity of $C([-1, 1])$, we have a natural identification of $C([-1, 1]^n)$ and the n -fold tensor product $C([-1, 1]) \otimes_{\max} \dots \otimes_{\max} C([-1, 1])$. Let $\pi_i : C([-1, 1]) \rightarrow \mathcal{B}(\mathcal{H})$ be the $*$ -representation mapping the identity function to $A_i, i = 1, \dots, n$. Since the ranges of $\pi_i, i = 1, \dots, n$, pairwise commute, there exists a $*$ -representation $\pi : C([-1, 1]^n) \rightarrow \mathcal{B}(\mathcal{H})$ such that $\pi(x_i) = A_i, i = 1, \dots, n$. It remains to let Φ be the restriction of π to $C(n)$. \square

Our next proposition places $NC(n)$ into the setting of group operator systems described in Section 2.

Notation 5.4. Let $*_n \mathbb{Z}_2 = \mathbb{Z}_2 * \dots * \mathbb{Z}_2$ be the free product of n copies of the group \mathbb{Z}_2 of order two.

Proposition 5.5. *Let $u_i = u_i^* \in C^*(*_n \mathbb{Z}_2)$ be the generator of the i -th copy of \mathbb{Z}_2 in $*_n \mathbb{Z}_2$ and $\mathbf{u}_n = \{u_1, \dots, u_n\}$. Then the unital linear map $\Psi : \mathcal{S}(\mathbf{u}_n) \rightarrow NC(n)$ defined by $\Psi(u_i) = h_i, 1 \leq i \leq n$, is a complete order isomorphism.*

Proof. Since each u_i is a self-adjoint unitary, it has spectrum $\{-1, 1\}$ and hence $-1 \leq u_i \leq 1$. Hence, by the universal property of $NC(n)$, the map Ψ^{-1} is completely positive. It hence suffices to show that Ψ is completely positive. To prove this, it is enough to show that if \mathcal{H} is any Hilbert space and $A_i \in \mathcal{B}(\mathcal{H})$ satisfy $A_i = A_i^*$ and $-I \leq A_i \leq I, 1 \leq i \leq n$, then there is a unital completely positive map $\gamma : \mathcal{S}(\mathbf{u}_n) \rightarrow \mathcal{B}(\mathcal{H})$ with $\gamma(u_i) = A_i, i = 1, \dots, n$.

With A_i as above, let

$$U_i = \begin{bmatrix} A_i & (I - A_i^2)^{1/2} \\ (I - A_i^2)^{1/2} & -A_i \end{bmatrix}$$

be the Halmos dilation of A_i . Then U_i is a self-adjoint unitary for each i . By the universal property of $C^*(*_n \mathbb{Z}_2)$, there is a unital $*$ -homomorphism $\pi : C^*(*_n \mathbb{Z}_2) \rightarrow \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ with $\pi(u_i) = U_i, 1 \leq i \leq n$. Composing π with the compression onto the first copy of \mathcal{H} yields a unital completely positive map $\gamma : C^*(*_n \mathbb{Z}_2) \rightarrow \mathcal{B}(\mathcal{H})$ with $\gamma(u_i) = A_i$, which completes the proof. \square

Propositions 2.2 and 5.5 give the following immediate corollary.

Corollary 5.6. *Up to a $*$ -isomorphism, we have that $C_e^*(NC(n)) = C^*(*_n\mathbb{Z}_2)$.*

There is at least one other representation of $NC(n)$ worth mentioning.

Proposition 5.7. *Let $\{u_1, \dots, u_n\}$ be the generators of \mathcal{S}_n . Then the linear map $\gamma : NC(n) \rightarrow \mathcal{S}_n$ defined by $\gamma(h_i) = (u_i + u_i^*)/2$ is a complete order isomorphism onto its range, which possesses a completely positive left inverse.*

Proof. Since $-1 \leq (u_i + u_i^*)/2 \leq 1$, the universal property of $NC(n)$ implies that the map γ is unital and completely positive. Conversely, since $\|h_i\| \leq 1$, the universal property of \mathcal{S}_n implies that there is a unital completely positive map $\psi : \mathcal{S}_n \rightarrow NC(n)$ with $\psi(u_i) = h_i$, $i = 1, \dots, n$. But then $\psi((u_i + u_i^*)/2) = h_i$, $i = 1, \dots, n$, and so ψ is a left inverse of γ . It follows that γ is a complete order isomorphism onto its range. \square

It is easy to show that the spectrum of $(u_i + u_i^*)/2$ in $C^*(\mathbb{F}_n)$ is $[-1, 1]$. However, by Corollary 5.6, the spectrum of h_i in $C_e^*(NC(n))$ is $\{-1, 1\}$. Hence, the C^* -algebra generated by the image of $NC(n)$ in $C^*(\mathbb{F}_n)$ is much larger than its enveloping C^* -algebra.

Proposition 5.8. *Let $\mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2$ be the direct sum of n copies of the group of order two, let $v_i = v_i^* \in C^*(\mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2)$ be the generator of the i -th copy of \mathbb{Z}_2 and let $\mathcal{S}(\mathbf{v}_n) = \text{span}\{1, v_i : 1 \leq i \leq n\}$. Then the unital map $\Psi : \mathcal{S}(\mathbf{v}_n) \rightarrow C(n)$ defined by $\Psi(v_i) = x_i$, $1 \leq i \leq n$, is a complete order isomorphism.*

Proof. The proof proceeds as that of the previous result. Since the elements v_i are selfadjoint, pairwise commuting and satisfy $-1 \leq v_i \leq 1$, we have that Ψ^{-1} is completely positive. To see that Ψ is completely positive, use the Halmos dilation on the coordinate functions x_i to obtain *pairwise commuting* self-adjoint unitaries $V_i \in \mathcal{M}_2(C([0, 1]^n))$. The universal property of $C^*(\mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2)$ shows that there exists a $*$ -homomorphism $\pi : C^*(\mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2) \rightarrow \mathcal{M}_2(C([0, 1]^n))$ with $\pi(v_i) = V_i$, $i = 1, \dots, n$. Since Ψ is a compression of π , we have that Ψ is completely positive. \square

In [12], the notion of a *coproduct* of two operator systems was introduced. If \mathcal{S} and \mathcal{T} are operator systems, the coproduct operator system $\mathcal{S} \oplus_1 \mathcal{T}$ is characterized by the following universal property: whenever \mathcal{U} is an operator system and $\phi : \mathcal{S} \rightarrow \mathcal{U}$ and $\psi : \mathcal{T} \rightarrow \mathcal{U}$ are unital completely positive maps, there exists a unique unital completely positive map $\theta : \mathcal{S} \oplus_1 \mathcal{T} \rightarrow \mathcal{U}$ such that $\theta(\iota_{\mathcal{S}}(x)) = \phi(x)$ for all $x \in \mathcal{S}$ and $\theta(\iota_{\mathcal{T}}(y)) = \psi(y)$ for all $y \in \mathcal{T}$, where $\iota_{\mathcal{S}}$ and $\iota_{\mathcal{T}}$ are the canonical inclusions of \mathcal{S} and \mathcal{T} , respectively, into $\mathcal{S} \oplus_1 \mathcal{T}$. It can easily be checked that the coproduct is an associative operation.

On the other hand, the C^* -algebra $C^*(*_n\mathbb{Z}_2)$ is canonically $*$ -isomorphic to the free product (involving n terms) $C^*(\mathbb{Z}_2) * \dots * C^*(\mathbb{Z}_2)$. Since $C^*(\mathbb{Z}_2)$ is $*$ -isomorphic to ℓ_2^∞ via the Fourier transform, it now follows from [12, Proposition 4.3] and Proposition 5.5 that $NC(n)$ is canonically order isomorphic to the coproduct (involving n terms) $\ell_2^\infty \oplus_1 \dots \oplus_1 \ell_2^\infty$.

Another consequence of [12, Theorem 4.8] is that $NC(2)$ is completely order isomorphic to the quotient operator system $\ell^\infty / \text{Span}\{(1, 1, -1, -1)\}$. More precisely, let $p_i = (1 + u_i)/2$, $q_i = (1 - u_i)/2$, so that p_i and q_i are the spectral projections for the self-adjoint unitary u_i corresponding to the eigenvalues 1 and -1 , respectively, $i = 1, 2$. We have the following fact.

Proposition 5.9 (Kavruk). *The linear map $\gamma : \ell_4^\infty \rightarrow \mathcal{S}(\mathbf{u}_2)$ defined by*

$$\gamma((a_1, a_2, a_3, a_4)) = 1/2[a_1p_1 + a_2q_1 + a_3p_2 + a_4q_2]$$

is a complete quotient map.

The following corollary describes strict positivity in $\mathcal{M}_n(NC(2))$.

Corollary 5.10. *Let $C_0, C_1, C_2 \in M_n$ be self-adjoint. Then $C_0 \otimes 1 + C_1 \otimes h_1 + C_2 \otimes h_2$ is strictly positive in $\mathcal{M}_n(NC(2))$ if and only if there exists $A \in \mathcal{M}_n$ such that $(C_0/2 \pm C_1 + A)$ and $(C_0/2 \pm C_2 - A)$ are strictly positive in \mathcal{M}_n .*

Proof. By Proposition 5.5, $NC(2) = \mathcal{S}(\mathbf{u}_2)$. Letting $h_i = u_i$, $i = 1, 2$, we have by Proposition 5.9 that $C_0 \otimes 1 + C_1 \otimes h_1 + C_2 \otimes h_2$ is strictly positive in $\mathcal{M}_n(NC(2))$ if and only if

$$\begin{aligned} &C_0 \otimes 1/2[p_1 + q_1 + p_2 + q_2] + C_1 \otimes (p_1 - q_1) + C_2 \otimes (p_2 - q_2) = \\ &(C_0/2 + C_1) \otimes p_1 + (C_0/2 - C_1) \otimes q_1 + (C_0/2 + C_2) \otimes p_2 + (C_0/2 - C_2) \otimes q_2 \end{aligned}$$

is strictly positive. On the other hand, the latter term is strictly positive if and only if it has a strictly positive pre-image in $\mathcal{M}_n(\ell_4^\infty)$. But any pre-image must be of the described form for some A in \mathcal{M}_n . \square

Recall that the (matrix ordered) dual of a finite dimensional operator system is again an operator system. Since a map $\phi : \ell_n^\infty \rightarrow \mathcal{M}_p$ is completely positive if and only if $\phi(e_i) \geq 0$ for all i , we see that the map that identifies ϕ with $(\phi(e_1), \dots, \phi(e_n)) \in \mathcal{M}_p(\ell_n^\infty)$ defines a complete order isomorphism between $(\ell_n^\infty)^d$ and ℓ_n^∞ . This is the context for the following result.

Proposition 5.11. *The matrix ordered dual $NC(2)^d$ of $NC(2)$ is completely order isomorphic to the operator subsystem*

$$\mathcal{V} = \{(a, b, c, d) : a + b = c + d\} \subset \ell_4^\infty$$

via the map that sends a functional $f : NC(2) \rightarrow \mathbb{C}$ to the vector $(f(p_1), f(q_1), f(p_2), f(q_2))$.

Proof. Since the map $\gamma : \ell_4^\infty \rightarrow NC(2)$ is a complete quotient map, the adjoint map $\gamma^d : NC(2)^d \rightarrow (\ell_4^\infty)^d = \ell_4^\infty$ is a complete order inclusion [7, Proposition 1.8]. It is easy to verify that \mathcal{V} is the range of γ^d and that $\gamma^d(f) = (f(p_1), f(q_1), f(p_2), f(q_2))$. \square

Remark 5.12. A class of operator spaces denoted $NSG(n, k)$, $n, k \in \mathbb{N}$, was considered in [10]. It is not difficult to see that $NSG(n, 2)$ coincides with the operator space dual of $NC(n)$. However, since these objects were studied as operator spaces, their operator system structure was not discussed in detail. Our emphasis, on the other hand, is on the operator system tensor product properties of non-commutative n -cubes and for this we need characterizations of the matrix ordered duals of the operator system $NC(n)$. This will be fully developed in the next section.

We conclude this section with a description of the matrix ordered dual $NC(n)^d$ for any $n \geq 2$, in the spirit of [7].

Proposition 5.13. *The matrix ordered dual $NC(n)^d$ of $NC(n)$ is completely order isomorphic to the operator subsystem \mathcal{W} of $\oplus_{k=1}^n \mathcal{M}_2$ given by*

$$\mathcal{W} = \left\{ \bigoplus_{k=1}^n \begin{pmatrix} a_{11}^k & a_{12}^k \\ a_{21}^k & a_{22}^k \end{pmatrix} : a_{ii}^k = a_{jj}^l, a_{12}^k = a_{21}^l, i, j = 1, 2, k, l = 1, \dots, n \right\}.$$

Proof. Recall the completely positive maps $\gamma : NC(n) \rightarrow \mathcal{S}_n$ and $\psi : \mathcal{S}_n \rightarrow NC(n)$ from Proposition 5.7, given by $\gamma(h_i) = (u_i + u_i^*)/2$ and $\psi(u_i) = h_i$, $i = 1, \dots, n$. If $Y \in \mathcal{M}_p(NC(n))$ is strictly positive, then so is $X = \gamma^{(p)}(Y) \in \mathcal{M}_p(\mathcal{S}_n)$. Recall that ψ is a left inverse of γ . Thus, $Y = \psi^{(p)}(X)$, which shows that ψ is a complete quotient map (Proposition 1.5), and so ψ^d is a completely order embedding of $NC(n)^d$ into \mathcal{S}_n^d .

Let $\mathcal{U} \subseteq \oplus_{k=1}^n \mathcal{M}_2$ be the operator system given by

$$\mathcal{U} = \left\{ \bigoplus_{k=1}^n \begin{pmatrix} a_{11}^k & a_{12}^k \\ a_{21}^k & a_{22}^k \end{pmatrix} : a_{ii}^k = a_{jj}^l, i, j = 1, 2, k, l = 1, \dots, n \right\}$$

and let $\theta : \mathcal{S}_n^d \rightarrow \mathcal{U}$ be the complete order isomorphism from [7, Theorem 4.4]. Let $\{F_j\}_{j=-n}^n$ be the basis of \mathcal{S}_n^d , dual to the basis $\{u_j\}_{j=-n}^n$. It follows from the proof of [7, Theorem 4.4] that θ maps F_0 to I and F_j to $\oplus_{k=1}^n A_k$, where $A_k = 0$ if $k \neq j$ and $A_j = \begin{pmatrix} 0 & 1/(n+1) \\ 0 & 0 \end{pmatrix}$, $j = 1, \dots, n$. The composition $\theta \circ \psi^d : NC(n)^d \rightarrow \mathcal{U}$ is a complete order embedding. Let $\{H_0, H_1, \dots, H_n\}$ be the basis of $NC(n)^d$, dual to the canonical basis $\{1, h_1, \dots, h_n\}$ of $NC(n)$. From the definition of ψ , we have that $\psi^d(H_i) = F_i + F_{-i}$, $i = 1, \dots, n$. It follows that the range of $\theta \circ \psi^d$ is \mathcal{W} . \square

6. TENSOR PRODUCTS OF NON-COMMUTATIVE CUBES

Although not stated in the language of operator systems, the calculations in [19, 20] of various bipartite correlation boxes amount to the calculation of various tensor products on $NC(n) \otimes NC(m)$. However, our general theory of operator system tensor products and our duality results give an alternate approach to this theory. We give our own derivation of these tensor results in the present section, while in the next section we introduce bipartite correlation boxes and explain how to translate results between the two theories. We begin with a general nuclearity result.

Proposition 6.1. *Let \mathcal{R} be an operator system. Then $\mathcal{R} \otimes_{\min} NC(1) = \mathcal{R} \otimes_{\max} NC(1)$.*

An indirect way to see this statement is to use the fact that $NC(1)$ is completely order isomorphic to ℓ_2^∞ , which is a nuclear C^* -algebra. However, our results yield a very short direct proof.

Proof. We have that $\mathcal{M}_n(\mathcal{R} \otimes_{\max} NC(1))_+ \subseteq_{\text{coi}} \mathcal{M}_n(\mathcal{R} \otimes_{\min} NC(1))_+$, so it will be sufficient to prove the converse inclusion. The identification $\mathcal{M}_n(\mathcal{R} \otimes_{\min} \mathcal{S})_+ = (\mathcal{M}_n(\mathcal{R}) \otimes_{\min} \mathcal{S})_+$ shows that it suffices to consider the case $n = 1$.

Given $r_1, r_2 \in \mathcal{R}$, we have that $r_1 \otimes 1 + r_2 \otimes h_1 \in [\mathcal{R} \otimes_{\min} NC(1)]_+$ if and only if $r_1 \pm r_2 \in \mathcal{R}_+$. Since $r_1 + r_2$ and $r_1 - r_2$ are in \mathcal{R}_+ and $1 + h_1$ and $1 - h_1$ are in $NC(1)_+$, it follows that

$$2[r_1 \otimes 1 + r_2 \otimes h_1] = (r_1 - r_2) \otimes (1 - h_1) + (r_1 + r_2) \otimes (1 + h_1) \in (\mathcal{R} \otimes_{\max} NC(1))_+$$

and the proof is complete. \square

The following result shows that the operator system $NC(n)$ satisfies the hypotheses of Theorem 2.5.

Lemma 6.2. *For every operator system \mathcal{R} we have that $\mathcal{R} \otimes_c NC(n) \subseteq_{\text{coi}} \mathcal{R} \otimes_{\max} C^*(*_n \mathbb{Z}_2)$. Thus, $NC(m) \otimes_c NC(n) \subseteq_{\text{coi}} C^*(*_m \mathbb{Z}_2) \otimes_{\max} C^*(*_n \mathbb{Z}_2)$ and therefore $NC(m) \otimes_c NC(n) = NC(m) \otimes_{\text{ess}} NC(n)$.*

Proof. The proof follows closely the argument given in [6, Lemma 4.1]. It suffices to show that if $\phi : \mathcal{R} \rightarrow \mathcal{B}(\mathcal{H})$ and $\psi : NC(n) \rightarrow \mathcal{B}(\mathcal{H})$ are unital completely positive maps with commuting ranges then ψ can be extended to a (unital) completely positive map $\tilde{\psi} : C^*(*_n \mathbb{Z}_2) \rightarrow \mathcal{B}(\mathcal{H})$ whose range commutes with the range of ϕ . This is done similarly to [6, Lemma 4.1]: we have that h_i is a selfadjoint unitary, and hence $\psi(h_i)$ is a selfadjoint operator with $-I \leq \psi(h_i) \leq I$. Write

$$w_i = \begin{bmatrix} \psi(h_i) & (1 - \psi(h_i))^{1/2} \\ (1 - \psi(h_i))^{1/2} & -\psi(h_i) \end{bmatrix}.$$

Then w_i is a selfadjoint unitary, and by the universal property of $C^*(*_n \mathbb{Z}_2)$, there exists a $*$ -homomorphism $\pi : C^*(*_n \mathbb{Z}_2) \rightarrow \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ such that $\pi(h_i) = w_i$. Letting $\tilde{\phi}(r) = \begin{bmatrix} \phi(r) & 0 \\ 0 & \phi(r) \end{bmatrix}$, we conclude that the ranges of π and $\tilde{\psi}$ commute. Now letting $\tilde{\psi}(x) = p\pi(x)p$, where $p = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, we conclude that the ranges of $\tilde{\psi}$ and ϕ commute and since $\tilde{\psi}$ extends ψ , we have shown the first claim.

The second claim follows by applying the first claim twice and using the identification $\mathcal{R} \otimes_c \mathcal{S} = \mathcal{S} \otimes_c \mathcal{R}$. \square

Theorem 6.3. *For every operator system \mathcal{R} we have that $\mathcal{R} \otimes_{\min} NC(2) = \mathcal{R} \otimes_c NC(2)$.*

Proof. It is a rather well-known fact that the group $\mathbb{Z}_2 * \mathbb{Z}_2$ is amenable (see Remark 6.4). Hence, $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ is nuclear and it follows that $\mathcal{R} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_2) = \mathcal{R} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$. Since $\mathcal{R} \otimes_{\min} NC(2) \subseteq_{\text{coi}} \mathcal{R} \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ and, by Lemma 6.2, $\mathcal{R} \otimes_c NC(2) \subseteq_{\text{coi}} \mathcal{R} \otimes_{\max} C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$, the result follows. \square

Remark 6.4. One way to see that $\mathbb{Z}_2 * \mathbb{Z}_2$ is amenable is to note that the subgroup $H = \{(g_1 g_2)^n : n \in \mathbb{Z}\}$ is isomorphic to \mathbb{Z} and $(g_1 g_2)^{-1} = g_2 g_1$. If we let \mathbb{Z}_2 act on H by the idempotent automorphism of conjugation by g_2 , then $\mathbb{Z}_2 * \mathbb{Z}_2$ is seen to be the semidirect product of H by \mathbb{Z}_2 . Now use the fact the semidirect products of amenable groups are amenable.

The following computational lemmas about the behaviour of max will be useful below.

Lemma 6.5. *Let \mathcal{R} and \mathcal{S} be finite dimensional vector spaces, $\{r_1, \dots, r_m\} \subseteq \mathcal{R}$ and $\{s_1, \dots, s_n\} \subseteq \mathcal{S}$. Let $(x_{i,j}) \in \mathcal{M}_p(\mathcal{R})$ and $(y_{i,j}) \in \mathcal{M}_p(\mathcal{S})$ and assume that $(x_{i,j}) = \sum_{k=1}^m A_k \otimes r_k$ and $(y_{i,j}) = \sum_{l=1}^n B_l \otimes s_l$, where $A_k, B_l \in \mathcal{M}_p$, $k = 1, \dots, m$, $l = 1, \dots, n$. Then*

$$\sum_{i,j=1}^n x_{i,j} \otimes y_{i,j} = \sum_{k=1}^m \sum_{l=1}^n \text{Tr}(A_k B_l^t) r_k \otimes s_l,$$

where Tr denotes the unnormalised trace.

Proof. By the bilinearity of the tensor products it is enough to consider the case where $(x_{i,j}) = (a_{i,j}r_k)$ and $(y_{i,j}) = (b_{i,j}s_l)$. But, in this case, $\sum_{i,j=1}^p (a_{i,j}r_k) \otimes (b_{i,j}s_l) = \text{Tr}(AB^t)r_k \otimes s_l$, and the proof is complete. \square

Lemma 6.6. *Let \mathcal{R} and \mathcal{S} be finite dimensional operator systems, $\{r_1, \dots, r_m\} \subseteq \mathcal{R}$ and $\{s_1, \dots, s_n\} \subseteq \mathcal{S}$ be linear bases, and $u = \sum_{k=1}^m \sum_{l=1}^n x_{k,l}r_k \otimes s_l \in \mathcal{R} \otimes \mathcal{S}$. If u is strictly positive in $(\mathcal{R} \otimes_{\max} \mathcal{S})_+$, then there exist $p \in \mathbb{N}$ and elements $U_1 = \sum_{k=1}^m A_k \otimes r_k \in M_p(\mathcal{R})_+$ and $U_2 = \sum_{l=1}^n B_l \otimes s_l \in M_p(\mathcal{S})_+$ such that $\text{Tr}(A_k B_l^t) = x_{k,l}$, $1 \leq k \leq m, 1 \leq l \leq n$.*

Proof. Suppose that u is strictly positive. By Lemma 1.7, there exist $p \in \mathbb{N}$, $U_1 = (\alpha_{i,j}) \in M_p(\mathcal{R})_+$ and $U_2 = (\beta_{i,j}) \in M_p(\mathcal{S})_+$ such that $u = \sum_{i,j=1}^p \alpha_{i,j} \otimes \beta_{i,j}$. Write $U_1 = \sum_{k=1}^m A_k \otimes r_k$ and $U_2 = \sum_{l=1}^n B_l \otimes s_l$ and use Lemma 6.5 to obtain the desired form of u . \square

Remark 6.7. It is not difficult to see that every element u of the form prescribed in Lemma 6.6 is necessarily positive; however, this fact will not be needed in the sequel.

We recall the operator system \mathcal{V} from Section 5:

$$\mathcal{V} = \{(a, b, c, d) \in \ell_4^\infty : a + b = c + d\} \subseteq \ell_4^\infty.$$

If $p \in \mathbb{N}$ then

$$\mathcal{M}_p(\mathcal{V}) = \left\{ \sum_{i=1}^4 X_i \otimes e_i : X_i \in \mathcal{M}_p(\mathbb{C}), i = 1, 2, 3, 4, X_1 + X_2 = X_3 + X_4 \right\}.$$

Moreover,

$$\mathcal{M}_p(\mathcal{V})_+ = \left\{ \sum_{i=1}^4 X_i \otimes e_i \in \mathcal{M}_p(\mathcal{V}) : X_i \in \mathcal{M}_p(\mathbb{C})_+, i = 1, 2, 3, 4 \right\}.$$

The following result gives a more concrete representation of the strictly positive elements of $\mathcal{V} \otimes_{\max} \mathcal{V}$.

Proposition 6.8. *Let $u = \sum_{i,j=1}^4 q_{i,j}e_i \otimes e_j$ be a strictly positive element of $\mathcal{V} \otimes_{\max} \mathcal{V}$. Then there exists p and matrices $X_i, Y_j \in \mathcal{M}_p(\mathbb{C})_+$ with $X_1 + X_2 = X_3 + X_4$ and $Y_1 + Y_2 = Y_3 + Y_4 = I$ such that $q_{i,j} = \text{Tr}(X_i Y_j)$, $i, j = 1, 2, 3, 4$.*

Proof. By Lemma 6.6, there exist $p \in \mathbb{N}$ and elements $U_1 = \sum_{i=1}^4 X_i \otimes e_i$ and $U_2 = \sum_{j=1}^4 Y_j^t \otimes e_j$ in $\mathcal{M}_p(\mathcal{V})_+$ such that $\sum_{i,j=1}^4 \text{Tr}(X_i Y_j^t)e_i \otimes e_j = \sum_{i,j=1}^4 q_{i,j}e_i \otimes e_j$. The fact that $U_1, U_2 \in \mathcal{M}_p(\mathcal{V})_+$ implies moreover that $X_i, Y_j \in \mathcal{M}_p(\mathbb{C})_+$ and $X_1 + X_2 = X_3 + X_4$, $Y_1 + Y_2 = Y_3 + Y_4$. Since $Y_i \in \mathcal{M}_p(\mathbb{C})_+$ if and only if $Y_i^t \in \mathcal{M}_p(\mathbb{C})_+$, we may and do replace Y_i by Y_i^t .

We thus have $q_{i,j} = \text{Tr}(X_i Y_j)$ with $X_i, Y_j \in \mathcal{M}_p(\mathbb{C})_+$ and $X_1 + X_2 = X_3 + X_4$, $Y_1 + Y_2 = Y_3 + Y_4$.

Let P be a positive invertible matrix such that $\sum_{j=1}^4 P^{-1}Y_j P^{-1} = 2E$ where E is the projection onto $\ker(Y_1 + Y_2)^\perp$ and set $\hat{Y}_i = P^{-1}Y_i P^{-1}$, $\hat{X}_i = P X_i P$. We have that $\hat{Y}_1 + \hat{Y}_2 = \hat{Y}_3 + \hat{Y}_4 = E$ and $\text{Tr}(X_i Y_j) = \text{Tr}(\hat{X}_i \hat{Y}_j) = \text{Tr}((E \hat{X}_i E)(E \hat{Y}_j E))$. After replacing \hat{X}_i and \hat{Y}_j with $E \hat{X}_i E$ and $E \hat{Y}_j E$, respectively, the same equations will hold. Diagonalizing E , we may regard the matrices $E \hat{X}_i E$ and $E \hat{Y}_j E$ as matrices of a smaller size. So no generality is lost in assuming that $\sum_{j=1}^4 Y_j = 2I$ or, equivalently, that $Y_1 + Y_2 = Y_3 + Y_4 = I$. \square

Lemma 6.9. *Let $p \in \mathbb{N}$ and $X_i, Y_j \in \mathcal{M}_p(\mathbb{C})_+$, $i, j = 1, 2, 3, 4$, with $X_1 + X_2 = X_3 + X_4$ and $Y_1 + Y_2 = Y_3 + Y_4 = I$. Set $q_{i,j} = \text{Tr}(X_i Y_j)$, $i, j = 1, 2$. For $a, c \in \{0, 2\}$, set $S_{a,c}(j, k) = \min_{b=0,2} \{ \sum_{i=1}^2 \sqrt{q_{b+i,a+j}} \sqrt{q_{b+i,c+k}} \}$. Then the following inequality holds:*

$$\text{Tr}(X_1 + X_2) \leq \min_{a,c} \sum_{j,k=1}^2 S_{a,c}(j, k).$$

Proof. Set $q = \text{Tr}(X_1 + X_2)$ for brevity. For all $a, b, c \in \{0, 2\}$ we have

$$\begin{aligned} q &= \text{Tr}((Y_{a+1} + Y_{a+2})(X_{b+1} + X_{b+2})(Y_{c+1} + Y_{c+2})) \\ &= \sum_{i,j,k=1}^2 \text{Tr}(Y_{a+j} X_{b+i} Y_{c+k}) = \sum_{i,j,k=1}^2 \text{Tr}((Y_{a+j} X_{b+i}^{1/2})(X_{b+i}^{1/2} Y_{c+k})) \\ &= \sum_{i,j,k=1}^2 \text{Tr}((X_{b+i}^{1/2} Y_{a+j})^* (X_{b+i}^{1/2} Y_{c+k})) \leq \sum_{j,k=1}^2 \sum_{i=1}^2 \|X_{b+i}^{1/2} Y_{a+j}\|_2 \|X_{b+i}^{1/2} Y_{c+k}\|_2. \end{aligned}$$

On the other hand, if X and Y are positive matrices with $Y \leq I$, then

$$\|X^{1/2} Y\|_2^2 = \text{Tr}(X^{1/2} Y^2 X^{1/2}) \leq \text{Tr}(X^{1/2} Y X^{1/2}) = \|X^{1/2} Y^{1/2}\|_2^2.$$

It follows that

$$q \leq \sum_{i,j,k=1}^2 \|X_{b+i}^{1/2} Y_{a+j}^{1/2}\|_2 \|X_{b+i}^{1/2} Y_{c+k}^{1/2}\|_2.$$

Since $\|X^{1/2} Y^{1/2}\|_2^2 = \text{Tr}(Y^{1/2} X Y^{1/2}) = \text{Tr}(XY)$ whenever X and Y are positive matrices, we conclude that

$$q \leq \sum_{j,k=1}^2 \sum_{i=1}^2 \sqrt{q_{b+i,a+j}} \sqrt{q_{b+i,c+k}}$$

whenever $b \in \{0, 2\}$. \square

Lemma 6.9 gives us a Bell type inequality. It has a form similar to the CHSH inequality of [5], except the appearance of square roots seems to be new.

Theorem 6.10. *Let $u = \sum_{i,j=1}^4 q_{i,j} e_i \otimes e_j \in (\mathcal{V} \otimes_{\max} \mathcal{V})_+$. For $a, c \in \{0, 2\}$, set $S_{a,c}(j, k) = \min \{ \sum_{i=1}^2 \sqrt{q_{b+i,a+j}} \sqrt{q_{b+i,c+k}} \}$. Then for $d \in \{0, 2\}$ the following inequality holds:*

$$\sum_{i,j=1}^2 q_{d+i,d+j} \leq \min_{a,c} \sum_{j,k=1}^2 S_{a,c}(j, k).$$

Proof. For any $\delta > 0$, the element

$$u + \delta 1 \otimes 1 = \sum_{i,j=1}^2 (q_{i,j} + \delta) e_i \otimes e_j$$

is strictly positive; by Lemma 6.9, there exist $p \in \mathbb{N}$ and matrices $X_i, Y_j \in \mathcal{M}_p(\mathbb{C})_+$ such that $X_1 + X_2 = X_3 + X_4$, $Y_1 + Y_2 = Y_3 + Y_4 = I$ and $q_{i,j} + \delta = \text{Tr}(X_i Y_j)$. The result now follows by observing that, for $d \in \{0, 2\}$,

$$\text{Tr}(X_1 + X_2) = \text{Tr}((X_{d+1} + X_{d+2})(Y_{d+1} + Y_{d+2})) = \sum_{i,j=1}^2 (q_{d+i,d+j} + \delta)$$

and letting $\delta \rightarrow 0$. \square

Theorem 6.11. $\mathcal{V} \otimes_{\min} \mathcal{V} \neq \mathcal{V} \otimes_{\max} \mathcal{V}$.

Proof. If we identify $\ell_4^\infty \otimes \ell_4^\infty$ with 4×4 matrices by the map $e_i \otimes e_j \rightarrow E_{i,j}$, then $\mathcal{V} \otimes \mathcal{V}$ is identified with the 4×4 matrices such that:

- (1) the first two terms in each row has the same sum as the last two terms;
- (2) the first two terms in each column has the same sum as the last two terms.

Let

$$Q = [q_{ij}]_{i,j=1}^4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix};$$

clearly, Q is in $(\mathcal{V} \otimes_{\min} \mathcal{V})_+ = (\mathcal{V} \otimes \mathcal{V}) \cap (\ell_4^\infty \otimes_{\min} \ell_4^\infty)_+$. The proof will be complete if we show that $Q \notin (\mathcal{V} \otimes_{\max} \mathcal{V})_+$.

We have that $\sum_{i,j=1}^2 q_{d+i,d+j} = 2$, for $d \in \{0, 2\}$. Now set $a = 0, c = 2$. Taking $b = 0$, we see that

$$0 \leq S_{0,2}(2, 1) \leq \sum_{i=1}^2 \sqrt{q_{i,2}} \sqrt{q_{i,3}} = 0 \text{ and } 0 \leq S_{0,2}(1, 2) \leq \sum_{i=1}^2 \sqrt{q_{i,1}} \sqrt{q_{i,4}} = 0.$$

On the other hand, taking $b = 2$ yields

$$0 \leq S_{0,2}(1, 1) \leq \sum_{i=1}^2 \sqrt{q_{2+i,1}} \sqrt{q_{2+i,3}} = 0 \text{ and } 0 \leq S_{0,2}(2, 2) \leq \sum_{i=1}^2 \sqrt{q_{2+i,2}} \sqrt{q_{2+i,4}} = 0.$$

This violates the inequalities of Theorem 6.10. \square

Corollary 6.12. $NC(2) \otimes_c NC(2) \neq NC(2) \otimes_{\max} NC(2)$.

Proof. By Proposition 5.11, $NC(2)^d = \mathcal{V}$. Assume that $NC(2) \otimes_{\min} NC(2) = NC(2) \otimes_{\max} NC(2)$; by taking duals, we have $NC(2)^d \otimes_{\max} NC(2)^d = NC(2)^d \otimes_{\min} NC(2)^d$, which contradicts Theorem 6.11. The proof is completed by using the fact that $NC(2) \otimes_{\min} NC(2) = NC(2) \otimes_c NC(2)$ (see Theorem 6.3). \square

Theorem 6.13 (Tsirelson). *For every $n, m \geq 2$, $NC(n) \otimes_c NC(m) \neq NC(n) \otimes_{\max} NC(m)$.*

Proof. This follows from the $n = 2$ case, similarly to the proof of Theorem 3.8 and using the fact that there exists a completely order isomorphic inclusion $\iota : NC(2) \rightarrow NC(k)$ and a unital completely positive map $\psi : NC(k) \rightarrow NC(2)$ whose composition is the identity on $NC(2)$. \square

Remark 6.14. This last result gives an alternate proof of a weaker result than Theorem 3.8. Namely, it shows that for every $n, m \geq 2$, $\mathcal{S}_n \otimes_c \mathcal{S}_m \neq \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$. To see this, let γ_n and ψ_n be the maps defined in Proposition 5.7. Assume that $\mathcal{S}_n \otimes_c \mathcal{S}_m = \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$. By functoriality, the maps $\gamma_n \otimes_c \gamma_m : NC(n) \otimes_c NC(m) \rightarrow \mathcal{S}_n \otimes_c \mathcal{S}_m$ and $\psi_n \otimes_{\max} \psi_m : \mathcal{S}_n \otimes_{\max} \mathcal{S}_m \rightarrow NC(n) \otimes_{\max} NC(m)$ are unital and completely positive. Thus,

$$(\psi_n \otimes_{\max} \psi_m) \circ (\gamma_n \otimes_c \gamma_m) : NC(n) \otimes_c NC(m) \rightarrow NC(n) \otimes_{\max} NC(m),$$

is a unital completely positive map and hence $NC(n) \otimes_c NC(m) = NC(n) \otimes_{\max} NC(m)$, a contradiction.

We record the following corollary whose proof follows closely that of Theorem 3.9.

Corollary 6.15. *We have that $C_e^*(NC(n) \otimes_{\max} NC(m)) \neq C_e^*(NC(n)) \otimes_{\max} C_e^*(NC(m))$ for every $n, m \geq 2$.*

Remark 6.16. The fact that \mathbb{F}_2 embeds in $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and the technique of the second author from [11, Theorem 5.3] shows that Kirchberg's Conjecture is equivalent to the identity $\mathcal{S}(u_3) \otimes_{\min} \mathcal{S}(u_3) = \mathcal{S}(u_3) \otimes_c \mathcal{S}(u_3)$ or, equivalently, to the identity $NC(3) \otimes_{\min} NC(3) = NC(3) \otimes_c NC(3)$. So determining further relations in this direction will be quite difficult. Tsirelson [19, 20] makes some claims that, if true, would imply that $[NC(m) \otimes_{\min} NC(n)]_+ = [NC(n) \otimes_c NC(m)]_+$, i.e., that these operator systems are equal at the ground level. However, one step in his proof remains unjustified and was later posted as a problem. See Fritz' paper [8] for a discussion.

It is natural to wonder about some other operator system tensor products. In particular [14] introduces three other tensor products that lie between \otimes_{\min} and \otimes_c . If one assumes that $\mathcal{S} \subseteq_{\text{coi}} \mathcal{B}(\mathcal{H})$ and $\mathcal{T} \subseteq_{\text{coi}} \mathcal{B}(\mathcal{K})$, then these are given by the identifications,

$$\mathcal{S} \otimes_{\text{el}} \mathcal{T} \subseteq_{\text{coi}} \mathcal{B}(\mathcal{H}) \otimes_{\max} \mathcal{T}, \quad \mathcal{S} \otimes_{\text{er}} \mathcal{T} \subseteq_{\text{coi}} \mathcal{S} \otimes_{\max} \mathcal{B}(\mathcal{K}),$$

and

$$\mathcal{S} \otimes_e \mathcal{T} \subseteq_{\text{coi}} \mathcal{B}(\mathcal{H}) \otimes_{\max} \mathcal{B}(\mathcal{K}).$$

Proposition 6.17. *Let $n, m \in \mathbb{N}$, then $NC(n) \otimes_{\min} NC(m) = NC(n) \otimes_{\text{el}} NC(m) = NC(n) \otimes_{\text{er}} NC(m)$.*

Proof. It is easily checked that $NC(n)$ and $NC(m)$ have the OSLLP property of [13, Definition 8.2]. Hence, by Theorem 8.1 and Theorem 8.5 of [13] together with the fact that the el and er tensors are identical modulo the flip, we have the claimed equalities. \square

Question 6.18. This leads us to ask:

- (1) Are any of the operator systems $NC(m) \otimes_{\min} NC(n)$, $NC(m) \otimes_e NC(n)$, and $NC(m) \otimes_c NC(n)$ equal?
- (2) What about the $C(m) \otimes C(n)$ cases?

7. BIPARTITE CORRELATION BOXES

In this section, we identify and discuss the relation of our results from Section 6 with quantum correlations studied in [3], [8], [19], [20], among others. Suppose that Alice and Bob perform an experiment in which Alice is given an input value x and produces an output value a , while Bob is given an input value y and produces an output value b . We assume that the possible values of the x, y, a, b are 0 and 1. Let $p_{a|x}^1$ be the probability that Alice returns the value a provided she is given the input x ; similarly, let $p_{b|y}^2$ be the probability that Bob returns the value b provided he is given the input y . These probabilities satisfy the following standard conditions: $p_{a|x}^1 \geq 0$, $p_{b|y}^2 \geq 0$, for all $a, b, x, y \in \{0, 1\}$, $p_{0|x}^1 + p_{1|x}^1 = 1$ for $x = 0, 1$, and $p_{0|y}^2 + p_{1|y}^2 = 1$ for $y = 0, 1$.

It is clear that the family $(p_{0|0}^1, p_{1|0}^1, p_{0|1}^1, p_{1|1}^1)$ is an element of the positive cone of the operator system \mathcal{V} defined in Section 5; moreover, every element of

\mathcal{V}_+ , after normalisation, can be written in such a form. We similarly have that $(p_{0|0}^2, p_{1|0}^2, p_{0|1}^2, p_{1|1}^2) \in \mathcal{V}_+$.

Let $p_{a,b|x,y}$ be the probability that the pair (a, b) is produced as an output by Alice and Bob, provided that Alice is given an input x and Bob is given an input y . A *bipartite correlation box* (which will be simply referred to by a *box*) is a table of probabilities of the form $(p_{a,b|x,y})_{a,b,x,y}$, viewed as an element of ℓ_{16}^∞ . The positivity conditions $p_{a,b|x,y} \geq 0$, $a, b, x, y \in \{0, 1\}$ are supposed to hold, as is the normalisation condition $\sum_{a,b=0}^1 p_{a,b|x,y} = 1$, $x, y \in \{0, 1\}$. In a “non-signaling” experiment, Alice and Bob are “not allowed to communicate,” which, in terms of the probability table, is expressed by requiring that

$$p_{a,0|x,0} + p_{a,1|x,0} = p_{a,0|x,1} + p_{a,1|x,1} = p_{a|x}^1, \text{ for all } a, x \in \{0, 1\},$$

$$p_{0,b|0,y} + p_{1,b|0,y} = p_{0,b|1,y} + p_{1,b|1,y} = p_{b|y}^2, \text{ for all } b, y \in \{0, 1\}.$$

We will assume that all boxes represent probability distributions of non-signaling experiments.

A box $(p_{a,b|x,y})_{a,b,x,y}$ is called *local* if there exists a probability distribution $(r(\lambda))_\lambda$ (that is, a finite family $(r(\lambda))_\lambda$ of non-negative real numbers with $\sum_\lambda r(\lambda) = 1$) and, for each λ , elements $p^k(\lambda) = (p_{0|0}^k(\lambda), p_{1|0}^k(\lambda), p_{0|1}^k(\lambda), p_{1|1}^k(\lambda)) \in \mathcal{V}_+$, $k = 1, 2$, normalised so that $p_{0|0}^k(\lambda) + p_{1|0}^k(\lambda) = 1$, $k = 1, 2$, such that

$$p_{a,b|x,y} = \sum_\lambda r(\lambda) p_{a|x}^1(\lambda) p_{b|y}^2(\lambda), \quad a, b, x, y \in \{0, 1\}.$$

Tsirelson [19] introduced *quantum* correlation boxes. These are the probability distributions $(p_{a,b|x,y})$ given by $p_{a,b|x,y} = \text{Tr}(\rho(A_x^a \otimes A_y^b))$, where A_x^a and A_y^b are positive operators acting on corresponding Hilbert spaces \mathcal{H}_x and \mathcal{H}_y such that $A_x^0 + A_x^1 = I$ and $A_y^0 + A_y^1 = I$ for all $x, y \in \{0, 1\}$, and ρ is a positive trace-class operator of unit trace. Tsirelson showed that these operators can be taken to act on a Hilbert space of dimension 2.

Following [3], we let \mathcal{P} be the set of all correlation boxes, \mathcal{L} be the closure of the set of all local correlation boxes, and \mathcal{Q} be the closure of the set of all quantum correlation boxes. Clearly, $\mathcal{L} \subseteq \mathcal{Q} \subseteq \mathcal{P}$ and each of these sets is convex.

In the sequel, we also identify the linear space $\mathcal{V} \otimes \mathcal{V}$ with \mathcal{M}_4 by the mapping sending $e_i \otimes e_j$ to E_{ij} . We denote by BS_4 the set of all bistochastic matrices in \mathcal{M}_4 , that is,

$$BS_4 = \left\{ (a_{i,j}) \in \mathcal{M}_4 : a_{k,l} \geq 0, \sum_{j=1}^4 a_{k,j} = \sum_{i=1}^4 a_{i,l} = 1, k, l = 1, 2, 3, 4 \right\}.$$

We then view \mathcal{P} as a subset of \mathcal{M}_4 in a natural way by identifying the family $(p_{a,b|x,y})_{a,b,x,y}$ with the matrix

$$\begin{bmatrix} p_{0,0|0,0} & p_{0,1|0,0} & p_{0,0|0,1} & p_{0,1|0,1} \\ p_{1,0|0,0} & p_{1,1|0,0} & p_{1,0|0,1} & p_{1,1|0,1} \\ p_{0,0|1,0} & p_{0,1|1,0} & p_{0,0|1,1} & p_{0,1|1,1} \\ p_{1,0|1,0} & p_{1,1|1,0} & p_{1,0|1,1} & p_{1,1|1,1} \end{bmatrix}.$$

Under this identification, \mathcal{P} is a convex subset of BS_4 .

To facilitate numbering, we re-label the generators of $NC(2)$ as $E_1 = p_1, E_2 = q_1, E_3 = p_2$, and $E_4 = q_2$, where $p_i = \frac{1+u_i}{2}$ and $q_i = \frac{1-u_i}{2}$, $i = 1, 2$. We note

that E_i , $i = 1, 2, 3, 4$, are positive operators satisfying the relations $E_1 + E_2 = E_3 + E_4 = I$. Similarly, we label the generators of the commutative operator system $C(2) = \text{Span}\{1, x_1, x_2\}$ by $f_1 = (1 + x_1)/2$, $f_2 = (1 - x_1)/2$, $f_3 = (1 + x_2)/2$, and $f_4 = (1 - x_2)/2$.

We recall from [18] that every ordered $*$ -vector space W with positive cone W_+ and Archimedean unit e can be equipped with a “maximal” operator system structure, that is, there exists a family $(C_n)_{n \in \mathbb{N}}$ of matrix cones such that $C_1 = W_+$ and $OMAX(W) = (W, (C_n)_{n \in \mathbb{N}}, e)$ is an operator system having the property that every positive map $\phi : W \rightarrow \mathcal{B}(H)$ is completely positive when as a map from $OMAX(W)$ into $\mathcal{B}(H)$. Similarly, there is a minimal operator system structure $OMIN(W)$ and, if W is a finite-dimensional operator system, then $OMAX(W^d) = OMIN(W)^d$.

The following theorem summarizes some of our results and can be thought of as a dictionary between the language of correlation boxes and that of tensor products of group operator systems.

Theorem 7.1. *We have the following identities:*

- (i) $\mathcal{P} = (\mathcal{V} \otimes_{\min} \mathcal{V})_+ \cap BS_4 = \{(s(E_i \otimes E_j)) : s \text{ is a state on } NC(2) \otimes_{\max} NC(2)\};$
- (ii) $\mathcal{Q} = (\mathcal{V} \otimes_{\max} \mathcal{V})_+ \cap BS_4 = \{(s(E_i \otimes E_j)) : s \text{ is a state on } NC(2) \otimes_{\min} NC(2)\};$
- (iii) $\mathcal{L} = (OMAX(\mathcal{V}) \otimes_{\max} OMAX(\mathcal{V}))_+ \cap BS_4 = \{(s(f_i \otimes f_j)) : s \text{ is a state on } C(2) \otimes_{\min} C(2)\}.$

Proof. (i) Since $\mathcal{V} \otimes_{\min} \mathcal{V} \subseteq_{\text{coi}} \ell_4^\infty \otimes_{\min} \ell_4^\infty \equiv \mathcal{M}_4$, we have that $(\mathcal{V} \otimes_{\min} \mathcal{V})_+$ is the set of matrices with non-negative entries satisfying $q_{i,1} + q_{i,2} = q_{i,3} + q_{i,4}$ and $q_{1,j} + q_{2,j} = q_{3,j} + q_{4,j}$, $i, j = 1, 2, 3, 4$. From this it follows that $\mathcal{P} = (\mathcal{V} \otimes_{\min} \mathcal{V})_+ \cap BS_4$. But by Proposition 5.11 and the fact that the maximal and the minimal tensor products are dual to each other, we have that $(q_{i,j}) \in (\mathcal{V} \otimes_{\min} \mathcal{V})_+$ if and only if $q_{i,j} = f(E_i \otimes E_j)$ for some positive functional $f : NC(2) \otimes_{\max} NC(2) \rightarrow \mathbb{C}$ and the first set of equalities follows.

(ii) Let $\mathcal{Q}_1 = \{(s(E_i \otimes E_j)) : s \text{ is a state on } NC(2) \otimes_{\min} NC(2)\}$, and let A_i and B_i , $i = 1, 2, 3, 4$, be positive operators acting on Hilbert spaces \mathcal{H} and \mathcal{K} , respectively, such that $A_1 + A_2 = A_3 + A_4 = I$ and $B_1 + B_2 = B_3 + B_4 = I$, and ρ be a positive operator of trace class on $\mathcal{H} \otimes \mathcal{K}$. The maps $\phi : NC(2) \rightarrow \mathcal{B}(\mathcal{H})$ and $\psi : NC(2) \rightarrow \mathcal{B}(\mathcal{K})$ given by $\phi(E_i) = A_i$ and $\psi(E_i) = B_i$, $i = 1, 2, 3, 4$, are unital and completely positive; hence, the tensor product map $\phi \otimes \psi : NC(2) \otimes_{\min} NC(2) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ is a unital completely positive map. Thus, the linear functional $\omega : NC(2) \otimes_{\min} NC(2) \rightarrow \mathbb{C}$ given by $\omega(u) = \text{Tr}(\rho(\phi \otimes \psi)(u))$, $u \in NC(2) \otimes_{\min} NC(2)$, is a state. It follows that all quantum correlation boxes are contained in \mathcal{Q}_1 ; since \mathcal{Q}_1 is a closed set, we conclude that $\mathcal{Q} \subseteq \mathcal{Q}_1$.

To show that $\mathcal{Q}_1 \subseteq \mathcal{Q}$, suppose that ω is a state of $NC(2) \otimes_{\min} NC(2)$. Represent the C^* -algebra $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ faithfully on a Hilbert space \mathcal{H} . Then ω has an extension to a state (denoted in the same way) of $C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. Since the latter C^* -algebra is separable, ω can be approximated pointwise by restrictions of normal states on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. We may therefore assume that ω is the

restriction to $C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ of a normal state on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. Hence, there exists a positive trace class operator ρ on $\mathcal{H} \otimes \mathcal{H}$ such that $\omega(u) = \text{Tr}(u\rho)$, $u \in NC(2) \otimes_{\min} NC(2)$. Letting $\iota : NC(2) \rightarrow C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ be the canonical inclusion, we see that $\omega(E_i \otimes E_j) = \text{Tr}((\iota(E_i) \otimes \iota(E_j))\rho)$, $i, j = 1, 2, 3, 4$. Thus, $(\omega(E_i \otimes E_j)) \in \mathcal{Q}$, and hence $\mathcal{Q} = \mathcal{Q}_1$.

To show that \mathcal{Q} coincides with $(\mathcal{V} \otimes_{\max} \mathcal{V})_+ \cap BS_4$, we need only take into account that $(NC(2) \otimes_{\min} NC(2))^d = \mathcal{V} \otimes_{\max} \mathcal{V}$.

(iii) Let $\mathcal{L}_1 = \{(s(f_i \otimes f_j)) : s \text{ is a state on } C(2) \otimes_{\min} C(2)\}$, and suppose that we are given a local box whose entries have the form

$$p_{a,b|x,y} = \sum_{\lambda} r(\lambda) p_{a|x}^1(\lambda) p_{b|y}^2(\lambda), \quad a, b, x, y \in \{0, 1\}.$$

Letting s_{λ}^1 and s_{λ}^2 be the states on $C(2)$ given by

$$(s_{\lambda}^k(f_1), s_{\lambda}^k(f_2), s_{\lambda}^k(f_3), s_{\lambda}^k(f_4)) = (p_{0|0}^k(\lambda), p_{1|0}^k(\lambda), p_{0|1}^k(\lambda), p_{1|1}^k(\lambda)), \quad k = 1, 2,$$

and $s = \sum_{\lambda} r(\lambda) s_{\lambda}^1 \otimes s_{\lambda}^2$, we have that $(p_{a,b|x,y})$ coincides with $(s(f_i \otimes f_j))$. It follows that $\mathcal{L} \subseteq \mathcal{L}_1$. Conversely, if s is any state on $C(2) \otimes_{\min} C(2)$, then s can be extended to a state on $C([-1, 1]) \otimes_{\min} C([-1, 1])$ and is hence given by integration against a probability measure on $[-1, 1] \times [-1, 1]$. But every such measure is in the weak* closed convex hull of the point evaluations, and it follows that $\mathcal{L}_1 \subseteq \mathcal{L}$; thus, $\mathcal{L} = \mathcal{L}_1$.

Now $(C(2) \otimes_{\min} C(2))^d = C(2)^d \otimes_{\max} C(2)^d$. Since $C(2) = \text{OMIN}(C(2))$, we have that $C(2)^d = \text{OMAX}(C(2)^d)$. Finally, the equality $C(2)^d = \mathcal{V}$ of ordered spaces implies the equality $C(2)^d = \text{OMAX}(\mathcal{V})$ as operator systems; hence, the proof of (iii) is complete. \square

Now that we see the identifications of images of states with sets of boxes we may apply Tsirelson's result [20] that $\mathcal{Q} \neq \mathcal{P}$ to deduce directly that $[NC(2) \otimes_{\min} NC(2)]_+ \neq [NC(2) \otimes_{\max} NC(2)]_+$.

Now that we see the connection between sets of boxes and tensor products more clearly it is interesting to ask the following questions:

Question 7.2. For each tensor product τ between min and max, we obtain a set of boxes between \mathcal{Q} and \mathcal{P} by considering the convex set

$$(\mathcal{V} \otimes_{\tau} \mathcal{V})_+ \cap BS_4.$$

What are the relationships among these for the tensor products that have already been introduced? Do any of these have physical meaning?

Dually, we could generate boxes by considering tensor products on $NC(2) \otimes NC(2)$. But since $NC(2) \otimes_{\min} NC(2) = NC(2) \otimes_c NC(2)$, we would only get new families of boxes by looking at tensors between \otimes_c and \otimes_{\max} . The tensors that have been introduced lie between \otimes_{\min} and \otimes_c , and so they generate no new families of boxes when applied to $NC(2)$.

Question 7.3. In a similar fashion one could look at tensors on either $C(2) \otimes C(2)$ or its dual to generate new families of boxes between \mathcal{L} and \mathcal{P} . What are the relationships here and do any have physical meaning?

REFERENCES

- [1] T. Ando. Structure of operators with numerical radius one. *Acta Sci. Math. (Szeged)*, 34:11–15, 1973.
- [2] W. Arveson. Subalgebras of C^* -algebras. II. *Acta Math.*, 128(3-4):271–308, 1972.
- [3] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu and D. Roberts. Non-local correlations as an information theoretic resource. *Phys. Rev. A* 71, 022101 (2005) (11 pages).
- [4] M. D. Choi and E. G. Effros. Injectivity and operator spaces. *J. Functional Analysis*, 24(2):156–209, 1977.
- [5] J. F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, Proposed experiment to test local hidden variable theories. *Phys. Rev. Lett.*, 23(15):880-884, 1969.
- [6] D. Farenick, A. S. Kavruk, and V. I. Paulsen. C^* -algebras with the weak expectation property and a multivariable analogue of Ando’s theorem on the numerical radius. *J. Operator Theory*, to appear.
- [7] D. Farenick and V. I. Paulsen. Operator system quotients of matrix algebras and their tensor products. *Math. Scand.*, to appear.
- [8] T. Fritz. Polyhedral duality in Bell scenarios with binary observables. *J. Math. Phys.*, 53(7):072202, 17, 2012.
- [9] M. Hamana. Injective envelopes of operator systems. *Publ. Res. Inst. Math. Sci.*, 15(3):773–785, 1979.
- [10] M. Junge, M. Navascues, C. Palazuelos, D. Perez-Garcia, V. Scholz and R. F. Werner. Connes’ embedding problem and Tsirelson’s problem *J. Math. Phys.* 52, 012102 (2011), 12 pages
- [11] A. S. Kavruk. Nuclearity related properties in operator systems. *preprint (arXiv:1107.2133)*, 2011.
- [12] A. S. Kavruk. The weak expectation property and Riesz interpolation. *preprint (arXiv:1201.5414)*, 2012.
- [13] A. S. Kavruk, V. I. Paulsen, I. G. Todorov, and M. Tomforde. Quotients, exactness and nuclearity in the operator system category. *preprint (arXiv:1008.281)*, 2010.
- [14] A. S. Kavruk, V. I. Paulsen, I. G. Todorov, and M. Tomforde. Tensor products of operator systems. *J. Funct. Anal.*, 261(2):267–299, 2011.
- [15] E. Kirchberg. On nonsemisplit extensions, tensor products and exactness of group C^* -algebras. *Invent. Math.*, 112(3):449–489, 1993.
- [16] E. Kirchberg and S. Wassermann. C^* -algebras generated by operator systems. *J. Funct. Anal.*, 155(2):324–351, 1998.
- [17] V. Paulsen. *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.
- [18] V. I. Paulsen, I. G. Todorov, and M. Tomforde. Operator system structures on ordered spaces. *Proc. London Math. Soc.* 102(1):25–49, 2011.
- [19] B. S. Tsirelson. Quantum generalizations of Bell’s inequality. *Lett. Math. Phys.*, 4(2):93–100, 1980.
- [20] B. S. Tsirelson. Some results and problems on quantum Bell-type inequalities. *Hadronic J. Suppl.*, 8(4):329–345, 1993.
- [21] M. E. Walter. Algebraic structures determined by 3 by 3 matrix geometry. *Proc. Amer. Math. Soc.*, 131(7):2129–2131, 2003.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF REGINA, REGINA, SASKATCHEWAN S4S 0A2, CANADA

E-mail address: `douglas.farenick@uregina.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801, U.S.A.

E-mail address: `kavruk@illinois.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77204-3476, U.S.A.

E-mail address: `vern@math.uh.edu`

PURE MATHEMATICS RESEARCH CENTRE, QUEEN’S UNIVERSITY BELFAST, BELFAST BT7 1NN, UNITED KINGDOM

E-mail address: `i.todorov@qub.ac.uk`